Discussion

Pricing of European Options Using Empirical Characteristic Functions
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1. Summary

1.1. Option Prices

- Log-returns X, with characteristic function $\phi_{X_T}(u) = E\left[e^{iuX_T}\right]$
- The Fourier transform relates the characteristic function and the density

- Call price at time 0 on an asset with price S_0 :

$$
\begin{aligned}
C(T,K) &= e^{-rT} E_0^Q\left[(S_T-K)^+\right] \\
&= S_0 - \frac{(S_0K)^{1/2}}{\pi} e^{-\frac{(r-w)T}{2}} \\
&\times \int_0^\infty \mathrm{Re}\left[e^{-iu\left(\ln\frac{S_0}{K}+(r+w)T\right)}\phi_{X_T}(-u-i/2)\right]\frac{du}{(u^2+1/4)}
\end{aligned}
$$

where $w = -(1/T)\ln\left(\phi_{X_T}(-i)\right)$.

- This formula is an inversion formula tailored for the payoff $(S_T-K)^+ = \left(e^{X_T}-K\right)^+$, just like the Fourier transform is tailored for the payoff $1_{\{S_T=s\}}$.

1.2. Estimating $\phi_{X_T}(\cdot)$

- Summing up log-returns
  - Option expiration is T, for example $T = 1$ year
  - Use higher frequency data, for example daily log-returns: $\Delta = 1$ day
  - If log-returns are iid, then $\phi_{X_T}(u) = \left[\phi_{X_\Delta}(u)\right]^p$ where $p = T/\Delta$.

- Estimating $\phi_{X_\Delta}(u)$ using daily returns $X_{j\Delta}$, $j = 1, \ldots, n$
  - Use the empirical characteristic function (ECF),

$$\hat{\phi}_{X_\Delta}(u) = \frac{1}{n} \sum_{j=1}^{n} e^{iuX_{j\Delta}}$$

    which is a sample estimator of $\phi_{X_\Delta}(u) = E\left[e^{iuX_\Delta}\right]$.

  - Standard convergence properties of $\hat{\phi}_{X_\Delta}$ to ϕ_{X_Δ}.

- Implementation: need an estimate of ϕ_{X_Δ}, p and w, then from the formula above get an option price $C_{w,p,\phi}(T, K)$.

1.3. Implementation

- Method 1: use the ECF $\hat{\phi}_{X_\Delta}$, use $p = T/\Delta$ to get to $\hat{\phi}_{X_T}$, plug $\hat{\phi}_{X_T}$ into w to get a $\hat{w}$ and compute the call pricing function $C_{w,p,\phi}(T, K)$

- Method 2: same except that p is obtained as

$$p^* = \arg\min_p \sum_{l=1}^{k} \left\{ C_{\hat{w},p,\hat{\phi}}(T, K_l) - C_{\text{market}}(T, K_l) \right\}^2 .$$

- Method 3: same except that w is obtained as

$$w^* = \arg\min_w \sum_{l=1}^{k} \left\{ C_{w,\hat{p},\hat{\phi}}(T, K_l) - C_{\text{market}}(T, K_l) \right\}^2 .$$

- Method 4: same except that (w, p) are obtained as

$$(w^*, p^*) = \arg\min_{(w,p)} \sum_{l=1}^{k} \left\{ C_{w,p,\hat{\phi}}(T, K_l) - C_{\text{market}}(T, K_l) \right\}^2 .$$

- Method 5: replace the ECF with the parametric formula corresponding to the Variance Gamma model. Minimize the parameters of the VG model as above.

2. Comments and Suggestions

2.1. From P to Q

- Empirical characteristic function $\hat{\phi}_{X_\Delta}(u) = \frac{1}{n}\sum_{j=1}^{n} e^{iuX_{j\Delta}}$ is estimated under P instead of Q since it is based on observed, unadjusted, log-returns $X_{j\Delta}$, $j = 1, \ldots, n$.

- So the expected value that is estimated is E_0^P, not E_0^Q.

- The adjustment is through the term w, which is designed to make the underlying asset price a martingale.

- w provides a mean adjustment.

- Would conceivably need additional degrees of freedom to parametrize the discrepancies between P and Q, especially over long horizons T.

- This is what Girsanov's Theorem provides.

- Markets can be incomplete when jumps are involved, so Q may not be unique here.

2.2. Adequacy of the Assumptions

- Log-returns are assumed to be iid, as implied by a Lévy process.

- Then they are summed up to go from Δ to T, which gives rise to the factor $p = T/\Delta$
  - Detail, $p = 252$ instead of 365
  - What is the meaning of not setting p to be the actual ratio of T over Δ?

- Most models based on semimartingales such as

$$\frac{dS_t}{S_{t-}} = \mu_{t-}dt + \sigma_{t-}dW_t + \delta_{t-}dL_t$$

will not imply iid log-returns even though both W (Brownian motion) and L (a pure jump Lévy process) are Lévy processes.

- In fact, this is already the case without a jump process, as in $\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t$.

- In any event, this is a testable hypothesis.

- Empirical facts regarding whether aggregating p log-returns to compute a longer horizon log-return: variance ratio tests, etc.

2.3. Evaluating the Five Methods

- Parameters $(w, p, \phi(\cdot))$ calibrated as described above

- Then performance of each method evaluated using

$$MAE = \sum_{l=1}^{k} \left| C_{\hat{w},\hat{p},\hat{\phi}}(T, K_l) - C_{\text{market}}(T, K_l) \right|$$

$$RMAE = \sum_{l=1}^{k} \frac{\left| C_{\hat{w},\hat{p},\hat{\phi}}(T, K_l) - C_{\text{market}}(T, K_l) \right|}{C_{\text{market}}(T, K_l)}$$

$$RMSE = \left[\sum_{l=1}^{k} \left| C_{\hat{w},\hat{p},\hat{\phi}}(T, K_l) - C_{\text{market}}(T, K_l) \right|^2 \right]^{1/2}$$

- Issues
  - This is done in-sample
  - In Methods 2-5, some of the parameters are already set by minimizing the MSE
  - The only difference betwen MSE and MAE is the greater emphasis on large deviations between market and model prices
  - So one would expect Methods 2-5 to have an unfair advantage over Method 1 (this is indeed what happens)
  - Would be better to calibrate the model on $[0, T_1]$ and examine the performance out-of-sample, on $[T_1, T_2]$.

2.4. Possible to Generalize?

- Lévy processes all have explicit characteristic functions, given by the Lévy-Khintchine formula

$$\psi(u) = \frac{1}{t} \ln \phi(u) = i\gamma u - \frac{\sigma^2}{2} u^2 + \int_{-\infty}^{+\infty} \left(e^{iuz} - 1 - \frac{izu}{1+z^2} \right) \nu(dz)$$

for all models subject to the condition of integrability of $\phi(-i)$ and the option price.

- Choice of martingale equivalent measure Q when markets are incomplete.

2.5. Minor Comments

- Why does the VG model play a special role among Lévy processes?

- Could add a comparison to the “practitioner’s model” which often serves as the benchmark in similar studies:
  - Calibrate a functional form to the implied volatility smile and plug into the Black-Scholes formula
  - And compare their out-of-sample performance.