

Binomial Approximations in Finance

John van der Hoek

**Discipline of Applied Mathematics,
University of Adelaide.**

Seminar at the Workshop
Stochastic Analysis and its Applications
in Finance and Insurance
University of Technology, Sydney
17th December 2007

Based on a chapter to appear in A.Bensoussan [ed]
“Mathematical Modeling and Numerical Methods in Fi-
nance”, Elsevier 2008.

Abstract

We present a novel way to approximate a diffusion process by a recombining binomial tree model. The method is obtained by approximating a procedure to find a weak solution of a stochastic differential equation.

If the original diffusion is expressed in risk neutral terms, then the binomial tree can be used to approximate the value of a wide number of financial derivatives and if the the original diffusion is expressed in real world probabilities, then the tree could be used to provide approximate simulations that could be used in risk analysis.

The Problem

We shall provide a recombining binomial tree model approximation for the solution of the stochastic differential equation:

$$\begin{aligned}dS(t) &= \mu(t, S(t))dt + \sigma(t, S(t))dB(t) \\ S(0) &= S\end{aligned}$$

on some probability space (Ω, \mathcal{F}, P) , where B is standard one-dimensional Brownian motion. We will study these equations over a time interval $[0, T]$.

Step 1: Construct a weak solution

On a probability space $(\Omega, \mathcal{F}, \bar{P})$ let \bar{B} be standard one-dimensional Brownian motion and suppose that

$$S(t) = \phi(t, \bar{B}(t)) \quad (1)$$

where ϕ solves the differential equation

$$\begin{aligned} \frac{\partial \phi}{\partial z}(t, z) &= \sigma(t, \phi(t, z)) \\ \phi(0, 0) &= S \end{aligned}$$

then

$$dS(t) = \bar{m}(t, \bar{B}(t))dt + \sigma(t, S(t))d\bar{B}(t)$$

where

$$\begin{aligned} \bar{m}(t, \bar{B}(t)) &= \frac{\partial \phi}{\partial t}(t, \bar{B}(t)) \\ &+ \frac{1}{2}\sigma(t, \phi(t, \bar{B}(t)))\frac{\partial \sigma}{\partial z}(t, \phi(t, \bar{B}(t))) \end{aligned}$$

We now change to a new probability P with the use of Girsanov transformation:

$$\frac{dP}{d\bar{P}} \Big|_{\mathcal{F}_T} = \exp \left(\int_0^T \psi(s) d\bar{B}(s) - \frac{1}{2} \int_0^T \psi(s)^2 ds \right)$$

If ϕ satisfies the Novikov condition, say, then

$$B(t) = \bar{B}(t) - \int_0^t \psi(s) ds$$

is standard Brownian motion under P when \bar{W} is standard Brownian motion under \bar{P} .

Under P, equation the dynamics of S becomes

$$dS(t) = \bar{m}(t, \bar{B}(t))dt + \sigma(t, S(t)) [dB(t) + \psi(t)dt]$$

We choose ψ so that

$$\mu(t, S(t)) = \bar{m}(t, \bar{B}(t)) + \sigma(t, S(t))\psi(t)$$

or

$$\psi(t) = \frac{\mu(t, \phi(t, \bar{B}(t))) - \bar{m}(t, \bar{B}(t))}{\sigma(t, \phi(t, \bar{B}(t)))} \equiv \Psi(t, \bar{B}(t))$$

Thus under P, (1) provides a (**weak**) solution of the original equation .

Step 2: Approximate this construction

Let N be a positive integer and set $\Delta t = \frac{T}{N}$.

We then define:

$$S(0, 0) = S$$

$$S(n, j) = \phi(n\Delta t, (2j - n)\sqrt{\Delta t})$$

for $j = 0, 1, \dots, n$ and $n = 0, 1, \dots, N$

From (n, j) (time n and state j) we can move to either $(n + 1, j + 1)$ or $(n + 1, j)$. In this way we obtain a recombining binomial tree of values for S . If $(n, j) \rightarrow (n + 1, j + 1)$ and $(n, j) \rightarrow (n + 1, j)$ occur with **equal probability**, then this recombining tree model provides a numerical approximation to the dynamics of S under \bar{P} . Nelson and Ramaswamy (1990) provides a proof of this claim.

We now assign new transition probabilities

$$p(n, j) \text{ to } S(n, j) \rightarrow S(n + 1, j + 1)$$

$$1 - p(n, j) \text{ to } S(n, j) \rightarrow S(n + 1, j)$$

so that the recombining tree model will provide an approximation to the dynamics of S under P .

The formula is:

$$p(n, j) = \frac{1}{2} + \frac{1}{2} \tanh[\Psi(n\Delta t, (2j - n)\sqrt{\Delta t})\sqrt{\Delta t}]$$

We now motivate this formula.

Let X be $\mathcal{F}_{t+\Delta t}$ measurable then

$$\mathbf{E}[X|\mathcal{F}_t] = \frac{\overline{\mathbf{E}}[X\Lambda_{t,t+\Delta t}|\mathcal{F}_t]}{\overline{\mathbf{E}}[\Lambda_{t,t+\Delta t}|\mathcal{F}_t]}$$

where

$$\Lambda_{t,\tau} = \exp\left(\int_t^\tau \psi(s)d\overline{B}(s) - \frac{1}{2}\int_t^\tau \psi(s)^2 ds\right)$$

and

$$\Lambda_{t,t+\Delta t} \approx \exp\left(\psi(t)\Delta\overline{B}(t) - \frac{1}{2}\psi(t)^2\Delta t\right)$$

With $X = I[\Delta\overline{B}(t) = \pm\sqrt{\Delta t}]$

$$\begin{aligned}\mathbf{E}[X|\mathcal{F}_t] &\approx \frac{\exp(\pm\psi(t)\sqrt{\Delta t})}{\exp(\psi(t)\sqrt{\Delta t}) + \exp(-\psi(t)\sqrt{\Delta t})} \\ &= \frac{1}{2}\left(1 \pm \tanh(\psi(t)\sqrt{\Delta t})\right)\end{aligned}$$

Examples

Example 1 (The Black and Scholes equation)

We have $\mu(t, x) = \mu x$ and $\sigma(t, x) = \sigma x$.

Then $\phi(t, z) = S \exp(\sigma z)$

and we set

$$\begin{aligned} S(n, j) &= S \exp[(2j - n)\sigma\sqrt{\Delta t}] \\ &= Su^j d^{n-j} \end{aligned}$$

where

$$\begin{aligned} u &= \exp[\sigma\sqrt{\Delta t}] \\ d &= \exp[-\sigma\sqrt{\Delta t}] \\ \psi(t) &= \frac{\mu}{\sigma} - \frac{1}{2}\sigma \end{aligned}$$

and

$$p(n, j) = \frac{1}{2} + \frac{1}{2} \tanh \left[\left(\frac{\mu}{\sigma} - \frac{1}{2}\sigma \right) \sqrt{\Delta t} \right]$$

Example 2

We have $\mu(t, x) = a - bx$ and $\sigma(t, x) = \sigma x$.

Then ϕ and $S(n, j)$ are as in Example 1.

But

$$\psi(t) = \frac{a}{\sigma S(t)} - \frac{b}{\sigma} - \frac{1}{2}\sigma$$

and

$$p(n, j) = \frac{1}{2} + \frac{1}{2} \tanh [\psi(n, j) \sqrt{\Delta t}]$$

where

$$\psi(n, j) = \frac{a}{\sigma S} \exp[(n - 2j)\sigma \sqrt{\Delta t}] - \frac{b}{\sigma} - \frac{1}{2}\sigma$$

Example 3 (The Cox-Ingersol-Ross equation)

We have $\mu(t, x) = a - bx$ and $\sigma(t, x) = \sigma\sqrt{x}$.

Then

$$\phi(t, z) \equiv \phi(z) = \begin{cases} [\sqrt{S} + \frac{1}{2}\sigma z]^2 & \text{if } \sqrt{S} + \frac{1}{2}\sigma z \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$S(n, j) = \phi((2j - n)\sqrt{\Delta t})$$

$$\psi(t) = \left(\frac{a}{\sigma} - \frac{\sigma}{2}\right) \frac{1}{\sqrt{S(t)}} - \frac{b}{\sigma} \sqrt{S(t)}$$

$$p(n, j) = \frac{1}{2} + \frac{1}{2} \tanh [\psi(n, j)\sqrt{\Delta t}]$$

where

$$\psi(n, j) = \left(\frac{a}{\sigma} - \frac{\sigma}{2}\right) \frac{1}{\sqrt{S(n, j)}} - \frac{b}{\sigma} \sqrt{S(n, j)}$$

Remark on Example 3

We note that as $S(n, j) \rightarrow 0+$

$$p(n, j) \rightarrow \begin{cases} 1 & \text{if } a > \frac{1}{2}\sigma^2 \\ 0 & \text{if } a < \frac{1}{2}\sigma^2 \end{cases}$$

which is why we often assume the second case ($\sigma^2 < 2a$) when S models an interest rate.

The CEV type model with $\sigma(t, x) = \sigma x^\beta$ where $0 < \beta < 1$ is treated in a similar way.

Example 4 (The Ornstein-Uhlenbeck Process)

Here we have $\mu(t, x) = \beta(a - x)$ and $\sigma(t, x) = \sigma$.

Then

$$\phi(t, z) \equiv \phi(z) = S + \sigma z$$

$$\psi(t) = \frac{\beta(a - S(t))}{\sigma}$$

$$S(n, j) = S + \sigma(2j - n)\sqrt{\Delta t}$$

$$p(n, j) = \frac{1}{2} + \frac{1}{2} \tanh \left[\frac{\beta(a - S(n, j))}{\sigma} \sqrt{\Delta t} \right]$$

and we note that

$$p(n, j) \quad \text{is} \quad \begin{cases} < \frac{1}{2} & \text{if } S(n, j) > a \\ > \frac{1}{2} & \text{if } S(n, j) < a \end{cases}$$

which supports the mean-reverting property.

The Vasicek interest model uses this process.

Example 5 (Ching+Li+Siu+Wu 2007)

[These authors priced real options of European and American type on this asset, solving a PDE by a finite difference scheme. Our scheme reproduced their results with a fraction of their effort.]

They have $\mu(t, x) = (b(t) - \kappa^\dagger \ln x)x$ and $\sigma(t, x) = \sigma x$ where $b(t) = \delta \sin(2\pi t + \xi) + \kappa\gamma + \frac{1}{2}\sigma^2$ and $\kappa^\dagger = \kappa + \bar{\lambda}\sigma$.

$$\begin{aligned} S(n, j) &= S \exp((2j - n)\sqrt{\Delta t}) \\ \psi(t) &= \frac{1}{\sigma} \left(b(t) - \kappa^\dagger \ln S(t) - \frac{1}{2}\sigma^2 \right) \\ p(n, j) &= \frac{1}{2} + \frac{1}{2} \tanh \left[\psi(n, j)\sqrt{\Delta t} \right] \end{aligned}$$

where

$$\psi(n, j) = \frac{1}{\sigma} \left(b(n\Delta t) - \kappa^\dagger \ln S(n, j) - \frac{1}{2}\sigma^2 \right)$$

Option Pricing

The discrete pricing formula is now:

$$V(N, j) = H(N, j) = g(S(N, j))$$

$$V(n, j) = \max(W(n, j), a \cdot E(n, j))$$

where

$$W(n, j) = \frac{p(n, j)V(n+1, j+1) + (1-p(n, j))V(n+1, j)}{e^{r\Delta t}}$$

$$H(0) = V(0, 0).$$

$a = 1$ for American style options, $a = 0$ for European style options.

For example

$$E(n, j) = (S(n, j) - S(0, 0)e^{rn\Delta t}) +$$

MATLAB code

```
function [price, lattice] = realoption2(S0,...
    K,r,kappa,gamma,lamdabar,kp,T,sigma, xi,...
    del,N,a)
deltaT = T/N;
lattice = zeros(N+1,1);
for j = 0:N
    lattice(j+1) = max(0, S0*exp(sigma*...
        (2*j-N)*sqrt(deltaT))-K*exp(r*T));
end for i =N-1:-1:0
    for j = 0:i
        p(j+1) = 0.5 + 0.5*tanh(...
            (del*sin(2*pi*i*deltaT+xi)+kappa*gamma...
            -kp*log(S0*exp(sigma*(2*j-i)*sqrt...
            (deltaT))))*sqrt(deltaT)/sigma);
        lattice(j+1) = max(a*(S0*exp(sigma*...
            (2*j-i)*sqrt(deltaT))...
            -K*exp(r*i*deltaT)),exp(-r*deltaT)*...
            (lattice(j+1) + ...
            (lattice(j+2) - lattice(j+1))*p(j+1)));
    end
end
price = lattice(1);
```

Remarks

This algorithm has a “replacing vector” form and performs faster for large choices of n .

$a = 1$ gives American style option prices;

$a = 0$ gives European style option prices.

Example 6 (Schwartz and Smith Model)

Using a notation similar to this paper we have

$$\begin{aligned}d\xi(t) &= (\mu_\xi - \lambda_\xi)dt + \sigma_\xi dB_\xi(t) \\d\chi(t) &= (-\kappa\chi(t) - \lambda_\chi)dt + \sigma_\chi dB_\chi(t)\end{aligned}$$

where μ_ξ , λ_ξ , σ_ξ , κ , λ_χ and σ_χ are constants and $dB_\xi(t)dB_\chi(t) = \rho dt$. We write

$$B_\chi(t) = \rho B_\xi(t) + \sqrt{1 - \rho^2} B_\xi^*(t)$$

where B_ξ and B_ξ^* are independent Brownian motions.

$$\begin{aligned}\xi(n, j, k) &= \xi(0) + \sigma_\xi(2j - n)\sqrt{\Delta t} \\ \chi(n, j, k) &= \chi(0) + \sigma_\chi[\rho(2j - n)\sqrt{\Delta t} \\ &\quad + \sqrt{1 - \rho^2}(2k - n)\sqrt{\Delta t}]\end{aligned}$$

and now for each node (n, j, k) we must calculate four probabilities:

$$p_1(n, j, k) \quad \text{for} \quad (n, j, k) \rightarrow (n + 1, j + 1, k + 1)$$

$$p_2(n, j, k) \quad \text{for} \quad (n, j, k) \rightarrow (n + 1, j + 1, k)$$

$$p_3(n, j, k) \quad \text{for} \quad (n, j, k) \rightarrow (n + 1, j, k + 1)$$

$$p_4(n, j, k) \quad \text{for} \quad (n, j, k) \rightarrow (n + 1, j, k).$$

Setting

$$\psi_1(n, j, k) = \frac{\mu_\xi - \lambda_\xi}{\sigma_\xi}$$

$$\psi_2(n, j, k) = \frac{-\kappa\chi(n, j, k) - \lambda_\chi - \rho\sigma_\chi\psi_1(n, j, k)}{\sqrt{1 - \rho^2\sigma_\chi}}$$

$$\tau_1(n, j, k) = \tanh \left[\psi_1(n, j, k) \sqrt{\Delta t} \right]$$

$$\tau_2(n, j, k) = \tanh \left[\psi_2(n, j, k) \sqrt{\Delta t} \right]$$

We use:

$$p_1(n, j, k) = \frac{1}{4}(1 + \tau_1(n, j, k))(1 + \tau_2(n, j, k))$$

$$p_2(n, j, k) = \frac{1}{4}(1 + \tau_1(n, j, k))(1 - \tau_2(n, j, k))$$

$$p_3(n, j, k) = \frac{1}{4}(1 - \tau_1(n, j, k))(1 + \tau_2(n, j, k))$$

$$p_4(n, j, k) = \frac{1}{4}(1 - \tau_1(n, j, k))(1 - \tau_2(n, j, k))$$

It is automatic in this construction that

$$p_i(n, j, k) \geq 0$$

for each $i = 1, 2, 3, 4$ and the probabilities sum to 1.

This could suggest that this algorithm is an improvement over another algorithm provided for this example by Hahn and Dyer (2004).

Conclusions

This approach can be applied to a wide range of examples, it is easy to code in MATLAB (say) to produce quick valuations even with $N=5000$ say. As well as providing a valuable pricing and simulation tool, it can be used as a benchmark to test other approaches to pricing.

It can be used in conjunction with some special techniques (like that developed by Hull and White) to estimate path dependent option prices.

The approach can be used to simulate processes for risk management. In that case the underlying model is provided with respect to real world probabilities.