Capital Market Equilibrium in a Mean-Weighted Lower Partial Moment Framework

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Abstract

While the EL-CAPM is the lower partial moment analogue of the Sharpe Lintner CAPM, the equivalent analogue of the Black CAPM has to date proven illusive. This paper introduces a new framework, the mean-weighted lower partial moment framework. A new form of linear separation obtaining between a zero-Beta asset and a risky portfolio is proven. From this separation result a new form of CAPM, linked with utility theory, is derived; the lower partial moment analogue of the Black CAPM. Unlike the Black CAPM the model cannot be rejected under a direct GMM test. A Bonferroni correction adds weight to this conclusion.

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I. Introduction

Variance is a dispersion measure that underpins much of asset pricing theory and risk modeling. Long before Value at Risk, variance was the key component in the CAPM pricing models of Sharpe (1964), Lintner (1965) and Black (1972). Variance has several properties that are mathematically very useful – the most important of which is the decomposition rule. It also has some well known deficiencies as a risk measure. Markowitz (1959) noted that variance is invariant under reflection; it fails to recognize asymmetry within asset return distributions. In his Nobel Prize lecture Markowitz (1992) suggested that semi-variance may serve as a two parameter objective function for portfolio construction that is more plausible than variance and less complicated than involving higher order moments. Furthermore variance considers favorable outcomes to be as important as adverse outcomes.

Unlike dispersion measures the lower partial moment (LPM) recognizes asymmetry within return distributions and does not involve favorable outcomes. This measure pertains only to adverse results, those below a defined threshold, rather than the entire return domain.

Bonds, equities and other assets have a limited liability feature which suggests that return distributions must be asymmetric. Theoretically, a corporate security can be viewed as a call option on the market value of the corporation's assets (Merton (1974)). More recently Brockman and Turtle (2003) proposed a barrier option framework for corporate security valuation[1]. Option frameworks for corporate security valuation, especially in the presence of corporate debt, have clear implications for the asymmetry of security returns. The LPM can be viewed as the (non-discounted) expected value of a powered put option on the returns of a portfolio. To use a risk or

[1] In this framework the corporation's value is a down-and-out call on the assets of the corporation. The knock-out is the debt level at which the corporation is bankrupt.

asset pricing framework that does not capture asymmetry is at least conceptually inconsistent with the way the values of individual firms are modeled. The risk literature has recognized the importance of distinguishing between adverse and favorable results and it has been a strong area of research. This has not been mirrored in asset pricing theory with the last significant theoretical contribution having taken place in the late 1990's (Grootveld and Hallerbach (1999)).

The psychological studies of Mao (1970), Unser (2000) and Veld and Veld-Merkoulova (2008) have supported the LPM over variance as a measure of the investor's perception of risk. These studies however do not point to clear determination of the LPM threshold. The thresholds considered are non-stochastic; the zero, the return of risk-free (zero-variance) asset, and the expected market return.

The LPM has a direct link with microeconomics. The relationship between stochastic dominance and the LPM under certain nested utility function classes was developed by several authors with the most complete undertaking found in Bawa (1975) and Bawa (1978). In those papers Bawa showed that the ordering of portfolios for an investor with certain forms of utility is equivalent to the ordering provided by the LPM of a corresponding degree (the power to which the moment is raised), evaluated for each threshold across the entire return domain. This relationship makes the LPM framework attractive for both asset pricing and portfolio optimization. Within portfolio optimization, it allows research to be conducted on determining which values the degree may adopt for individual investors of varying risk averseness. Within asset pricing it allows for research to be focused on determining a value for the degree of the representative investor.

In the 1970's calculation and optimization complexity presented major obstacles to applications of the first mean-lower partial moment (EL) asset pricing models of Hogan and Warren (1974) and Bawa and Lindenberg (1977). This optimization complexity arises from the lack of a decomposition rule[2] and thus the requirement for analysis across the entire joint distribution of asset returns. Modern computing and optimization algorithms have significantly reduced this problem. If the distribution of the portfolio's returns is assumed to belong to a distribution family then in some cases the LPM can be calculated directly, using recursive formulas or by using the partial moment generating function (Winkler, Roodman and Britney (1972))[3]. However such assumptions often reduce the EL-CAPM to the classical CAPM (Chow and Denning (1994)).

In this paper a new measure, the threshold probability weighted lower partial moment (TPWL) is introduced. The TPWL is a discerning and well founded measure of risk as: (i) it accommodates asymmetric return distributions, (ii) a special case, the LPM, appears to be indirectly supported by psychological studies (the lack of clarity on the threshold gives cause to believe that at least similar support would be shown), and (iii) it has a direct and rich link with utility theory.

Several properties of the TPWL (risk aversion, stochastic dominance, psychological studies and consistency with option based corporate valuation) motivate a key assumption on preferences; that investors will discern between investment choices using TPWL efficiency.

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[2] Within a mean variance framework the relationship $Var\left(\sum_{k=1}^{S} X_k R_k\right) = \sum_{i=1}^{S}\sum_{j=1}^{S} X_i X_j Cov\left(R_i, R_j\right)$ for a portfolio of S securities with weights X_i *(i=1,2,…,S)* is utilized in asset pricing. This relationship is the basis of the variance-covariance matrix. There is no equivalent decomposition rule within the LPM framework; $LPM_n\left(\sum_{k=1}^{S} X_k R_k, h\right) \neq \sum_{i=1}^{S}\sum_{j=1}^{S} X_i X_j CLPM_n\left(R_i, R_j, h\right)$.

[3] One example is the Pearson family which includes the normal, beta, gamma, chi-square and Student-t distributions.

The mean-threshold probability weighted lower partial moment (ET) framework is introduced for the purpose of asset pricing. The EL framework utilized by Hogan and Warren (1974) and Bawa and Lindenberg (1977) is a special case of the ET framework. However I assert that the ET framework does not possess the critical deficiency of the EL framework; the absence of a framework consistent portfolio separation result. Furthermore the ET-CAPM has a more general link (partial stochastic dominance) with utility theory than the EL-CAPM (singular stochastic dominance).

Two new results on portfolio separation in the ET framework are presented. The first result, zero-β separation, involves separation between the zero-β asset and a risky portfolio. Monetary separation is shown to be a special case of the first result. A second result, mean separation, is also derived in the presence of the zero-β. Mean separation has been a great source of confusion within the EL literature – the confusion having remained without clarification for a decade until Grootveld and Hallerbach (1999). This previous form of mean separation is also shown to be a special case.

Building on the zero-β separation result, a new partial equilibrium pricing model, the ET-CAPM, is derived. The ET-CAPM makes no distributional assumptions (beyond LPM existence) for the risky assets and no distributional assumptions regarding the zero-β asset. The EL-CAPM of Hogan and Warren (1974) and Bawa and Lindenberg (1977) are special cases of the ET-CAPM.

The pricing kernel (stochastic discount factor) for the ET-CAPM is derived from the utility function implied by the Lagrangian. This pricing kernel cannot be negative and thus there is an absence of arbitrage opportunities. Using a new result on the ratios of covariance, co-lower partial moments and expected returns the ET-CAPM can be derived.

In contrast to the mean-variance CAPM of Sharpe (1964) and Lintner (1965), the ET-CAPM has no requirement for multivariate elliptical distributions or quadratic utility. In the special case of elliptically distributed returns and the presence of a zero-variance asset this model reduces to the mean-variance CAPM.

This paper is structured as follows. Section II provides definitions for the LPM and TPWL, and a proof of TPWL convexity. A theorem relating stochastic dominance to the TPWL is derived. This theorem motivates the model developed later in Section V. Section III considers the notion of a risk-free asset in different frameworks and portfolio separation without a zero-variance asset. This discussion underlies an argument made in Section IV. Section IV focuses on three forms of portfolio separation in the ET framework. Earlier EL results are shown to be special cases. In Section V the ET-CAPM is derived from one of the separation results. The features of this model are discussed along with its connection with previous asset pricing models. In section VI the pricing kernel for the ET-CAPM is determined and verified. In Section VII the model is considered under divergent borrowing and lending facilities. Such a consideration is lacking in the EL literature. Section VIII contains a demonstration of the model which emphasizes an important distinction between this and earlier models. This is followed by an empirical test of the ET-CAPM and Black CAPM using the generalized method of moments in Section IX. At the 5% significance level the ET-CAPM is not rejected while the Black CAPM is. Lastly, Section X concludes the paper.

II. LPM & TPWL Definitions and Stochastic Dominance

In this section the LPM and the TPWL are defined. A theorem on TPWL convexity is derived that will be used in the derivation of the asset pricing model in section V. The relationship between the TPWL and stochastic dominance through investor utility is developed here; this is a source of economic richness within the model derived later. This reflects the relationship between the LPM and stochastic dominance as found in Bawa (1975) and can be viewed as a generalization of that work.

Definition (LPM): The n^{th} order lower partial moment (LPM) functional for a portfolio p is defined as:

$$LPM_n(R_p;h) = \int_{-\infty}^{h} (h - R_p)^n f(R_p) dR_p \quad [1]$$

Where: $f(R_p)$ is the probability distribution function for the returns R_p of portfolio p.

h is the threshold[4] rate about which the partial moment is calculated.

n is the order of the LPM_n.

Equivalently, this can be written as[5]:

$$LPM_n(R_p;h) = \int_{-\infty}^{+\infty} (h - R_p)^{+^n} f(R_p) dR_p = E\left[(h - R_p)^{+^n}\right] \quad [2]$$

The LPM_n can be normalized as a risk measure:

$$LPM_n^{1/n}(R_p;h) = \left(LPM_n(R_p,h)\right)^{1/n} \quad [3]$$

[4] To avoid confusion I use the term 'threshold' as opposed to 'target' (Hogan and Warren (1972)), 'disaster level' (Roy (1952) & Stone (1973)), or 'minimum acceptable return' (Lintner (1965)) as seen throughout the literature. These terms are suggestive of the investor's risk/return preferences and objectives (and reflect the assumed economic motivation for the investor) whereas 'threshold' is innocuous and broad enough to capture all these ideas.

[5] For brevity I use the notation $(X)^+$ for $\max(X, 0)$.

A. Stochastic Dominance and the LPM

Stochastic dominance provides a partial ordering of investment choices when investors possess concave utility functions whose derivatives exist and alternate in sign up to the n^{th} derivative. Consider the nested classes of utility functions defined as:

$$U_{n+1} = \left\{ u(x) \in U_n \middle| \, n = 0,1,2,3,\ldots \quad -\infty < (-1)^{n+1} u^{(n+1)}(x) < 0, \quad \forall x \in [a,b], \quad b > a \right\}$$

where $u^{(n+1)}$ is the $(n+1)^{\text{th}}$ derivative of the utility function, $u(x)$, assumed $(n+1)$-times differentiable, and U_0 is the set of all utility functions.

In a series of theorems Bawa (1976) showed that the ordering of portfolios of expected return μ for an investor with utility of class U_{n+1} is equivalent to the inverse ordering provided by the LPM of degree n across the return domain $[a,b]$.

> *Theorem 1 (Stochastic Dominance of Order* $n+1$*, Bawa):* Consider two cumulative return distributions F_A and F_B with a combined domain $[a,b]$. For all investors with utility functions in U_{n+1}, F_A is preferred to F_B, $F_A \underset{n+1}{\succcurlyeq} F_B$, if and only if:

$$\int_{-\infty}^{h} \int_{-\infty}^{x} \cdots \int_{-\infty}^{x} [F_B(x) - F_A(x)] \, dx^n \geq 0 \quad \forall h \in [a,b] \qquad [5a]$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{x} \cdots \int_{-\infty}^{x} [F_B(x) - F_A(x)] \, dx^k \geq 0 \quad k = 1,2,\cdots,n-1 \qquad [5b]$$

This can be expressed in terms of lower partial moments. The LPM of degree n can be integrated by parts n times to generate the n-fold integral of the cumulative distribution function multiplied by n-factorial:

$$LPM_n(R\,; h) = n! \int_{-\infty}^{h} \int_{-\infty}^{R} \cdots \int_{-\infty}^{R} F(R) dR^n$$

So equivalently, F_A stochastically dominates F_B in the $n+1^{\text{th}}$ degree if:

$$LPM_n(R_B;h) - LPM_n(R_A;h) \geq 0 \quad \forall h \in [a,b] \qquad [6a]$$

$$LPM_k(R_B;b) - LPM_k(R_A;b) \geq 0 \qquad k = 1,2,\cdots,n-1 \qquad [6b]$$

This result ($n=2$) motivated the EL-CAPM of Bawa and Lindenberg (1977) however their model did not incorporate $\forall h \in [a,b]$. Their model incorporated an LPM with a single value of h, namely the zero-variance risk free rate and thus utilized a necessary but not sufficient condition for stochastic dominance.

B. The Threshold Weighted Lower Partial Moment

Definition(TPWL): The n^{th} order threshold probability weighted lower partial moment (TPWL) functional for a portfolio p is defined here as:

$$TPWL_n(R_p, h; a, b) = \int_a^b \left[\int_{-\infty}^{h} (h - R_p)^n c(h, R_p) dR_p \right] dh \qquad [7]$$

where h is the stochastic threshold rate about which the partial moment is calculated.

a and b are the lower and upper bounds for the threshold h.

$c(h, R_p)$ is the joint probability distribution function for the returns R_p of portfolio p and h.

n is the degree of the $TPWL_n$.

In this definition the probability distribution function $c(h,R)$ weights the LPM contribution to the $TPWL_n$. If h is independent of R_p then their joint distribution can be written as $c(h,R_p) = f(R_p)g(h)$ to form:

$$TPWL_n(R_p, h; a, b) = \int_a^b \left[\int_{-\infty}^h (h - R_p)^n f(R_p) dR_p\right] g(h) dh \qquad [8]$$

The $TPWL_n$ can be normalized:

$$TPWL_n^{1/n}(R_p, h; a, b) = \left(\int_a^b \left[\int_{-\infty}^h (h - R_p)^n f(R_p) dR_p\right] g(h) dh\right)^{\frac{1}{n}} \qquad [9]$$

Which can also be written in terms of lower partial moments as:

$$TPWL_n^{1/n}(R_p, h; a, b) = \left(\int_a^b LPM_n(R_p; h) g(h) dh\right)^{\frac{1}{n}}$$

I now demonstrate the convexity of the $TPWL_n$ and thus the $TPWL_n^{1/n}$ for $n \geq 1$. This mirrors the LPM_n convexity proof found in Bawa (1978). Firstly we express the integral as the following expectation:

$$TPWL_n(R_p, h; a, b) = E_{f,g}\left[(h - R_p)^{+n}\right] \qquad [10]$$

Let **R** represent the vector of security returns and let the number of available assets be l. I define C to be a convex and compact choice set of feasible real valued ($\mathbb{R}$) weights, $\mathrm{C} \subset \mathbb{R}^l$. Let the portfolio weights be represented by $\mathbf{x}^{\mathrm{T}} \in \mathrm{C}$, where $\mathbf{x}^{\mathrm{T}}\mathbf{1} = 1$. An affine function of a convex set is itself convex. Thus $(h - R_p) = (h - \mathbf{x}^{\mathrm{T}}\mathbf{R})$ is a convex function of the portfolio weights $\mathbf{x}^{\mathrm{T}}$. Now

$$\frac{TPWL_n(R_p, h; a, b)}{dR_p} = E_g\left[(h - R_p)^n f(R_p)\right] \geq 0 \qquad R_p \in (-\infty, h) \qquad \forall n \geq 1 \qquad [11]$$

So the argument of the expectation operator, $E_{f,g}$, is monotone decreasing (convex) in R_p, $R_p \in (-\infty, h)$, for $n \geq 1$. From the linearity of the Expectations operator, the $TPWL_n$ is also convex in the portfolio weights, $\mathbf{x}^{\mathrm{T}}$; an affine function of a convex function is itself convex. Note lastly that $TPWL_n^{1/n}$ is also convex; $f(y) = y^{1/n}$ is convex and again a convex function of a convex function is itself convex. The fact will be utilized when forming a tangent portfolio to the mean-$TPWL_n$ efficient frontier. The convexity of the $TPWL_n$ means that the optimization problem is well behaved.

> *Theorem 2 (TPWL Convexity)*: Given a compact choice set C, for every $n \geq 1$, $TPWL_n$ is a convex function of $\mathbf{x}^{\mathrm{T}} \in \mathrm{C}$.

C. Stochastic Dominance and the TPWL

Here I present a new theorem for stochastic dominance in the ET framework equivalent to the EL theorems of Bawa.

> *Theorem 3 (Stochastic Dominance of Order* $n+1$*)*: Consider two cumulative return distributions, F_A and F_B, with a combined domain $[a, b]$. For all investors with utility functions in U_{n+1}, distribution F_A is preferred to distribution F_B, $F_A \underset{n+1}{\succcurlyeq} F_B$, if and only if:

$$\int_{a^*}^{b^*} [LPM_n(R_B; h) - LPM_n(R_A; h)] dh \geq 0 \quad for\ \forall\ [a^*, b^*] \subseteq [a, b] \qquad [12a]$$

$$LPM_k(R_B; b) - LPM_k(R_A; b) \geq 0 \quad k = 1, 2, \cdots, n-1 \qquad [12b]$$

In order to verify that F_A dominates F_B for a particular class of utility U_{n+1} the $TPWL_n$ function must be compared for all subintervals $[a^*, b^*] \subseteq [a, b]$. A necessary (but not sufficient) condition can be found by selecting an arbitrary subinterval, $[a^*, b^*] \subseteq [a, b]$ for which:

$$\int_{a^\circ}^{b^\circ} [LPM_n(R_B;h) - LPM_n(R_A;h)]dh \geq 0 \qquad for\ \forall\ [a^\circ, b^\circ] \subseteq [a^*, b^*] \qquad [13]$$

The probability density function (always positive) for the threshold can be placed within the integral without contravening the inequality to yield:

$$\int_{a^\circ}^{b^\circ} [LPM_n(R_B;h) - LPM_n(R_A;h)]g(h)dh \geq 0 \qquad for\ \forall\ [a^\circ, b^\circ] \subseteq [a^*, b^*] \qquad [14]$$

Which can be rewritten as:

$$TPWL_n(R_B, h; a^\circ, b^\circ) - TPWL_n(R_A, h; a^\circ, b^\circ) \geq 0 \qquad for\ \forall\ [a^\circ, b^\circ] \subseteq [a^*, b^*] \qquad [15]$$

This last result, a necessary but not sufficient condition for stochastic dominance, motivates the ET-CAPM in the same fashion that a singular LPM motivated the EL-CAPM of Bawa and Lindenberg (1977). Within this paper, instances where F_A dominates F_B for a particular class of utility U_N over a particular subinterval $[a^*, b^*]$ of the entire return domain $[a, b]$ will be termed cases of 'partial stochastic dominance'.

III. Equilibrium without a Zero-Variance Asset

In the EV (expected return – variance) framework of Sharpe (1964) and Lintner (1965) a security with a return variance of zero is considered risk free. In the EV framework of Black (1972) a zero-beta (zero-covariance) portfolio is utilized.

The requirements for a risk-free asset are far less restrictive in the EL framework; any asset with zero probability of yielding a return below the investor-defined threshold is risk free. Therefore the return of a risk free asset in the EL framework can be stochastic (non-zero variance) provided there is zero probability that these returns will fall below the given threshold.[6]

Hogan and Warren (1974) originally argued that the threshold in their EL-CAPM be the largest risk free rate under the LPM_2 measure. They subsequently assumed that such an asset exists and that its return is precisely h with probability one – in other words they assumed that the threshold rate is simply a zero variance return. Throughout the EL literature the risk free asset has been assumed to be the zero-variance asset. This may be because the roots of the EL-CAPM rest in the EV-CAPM and the Safety First measure of Roy (1952). Mathematically, the assumption that the zero-variance asset is the investor's threshold is convenient as it can be removed from the argument of the expectations operator and separation follows. Separation does not, however, obtain between an asset with a lower partial moment of zero and a risky portfolio in the EL framework. It seems reasonable to expect that in any valid framework separation should obtain between the zero risk and risky portfolios. The EL-CAPM therefore seems to be somewhat inconsistent with its origins.

[6] A simple example: when using monthly asset returns the US 30 day Treasury Bill rate is typically used as the zero variance rate over a one month horizon. However the strategy of investing overnight at the Federal Funds rate acts as an asset with zero LPM for an appropriately selected threshold .

In the framework introduced in this paper, the ET (Expected return – Threshold probability weighted LPM) framework, separation obtains between a zero-β asset and a risky portfolio. A zero-β asset is zero-$TPWL_n$ if the threshold is the stochastic return of the zero-β asset ($h = R_z$) and the lower and upper bounds (a, b) on $TPWL_n$ are that also of the zero-β asset;

$$TPWL_n(R_z, h = R_z; a, b) = \int_a^b \left[\int_{-\infty}^{R_z} (h - R_z)^n g(R_z) dR_z\right] dR_z = 0 \qquad [16]$$

It follows from arbitrage arguments[7] that there can be only one zero-variance asset in an EV or EL framework. For example, when working with monthly US data and a one month investment horizon, the adopted zero-variance asset is the 30-day Treasury bill. Within the Black formulation, a portfolio of treasuries is selected to provide the minimum variance zero-β portfolio. The same could be used for the ET framework however if investors are concerned with minimizing tail risk it seems reasonable that they would price with reference to an uncorrelated asset that had some form of lower bound. A likely candidate would seem to be the monthly returns generated from investing at the daily effective federal funds rate. Both the Black CAPM and ET-CAPM can be applied to economies that may not possess a viable zero-variance asset.

[7] With the suspension of credit issues.

IV. Zero-β, Monetary and Mean Separation in the EL and ET Frameworks

In the sub-sections that follow two new theorems on two fund separation within the ET framework are presented and the EL separation results (monetary and mean) are shown to be special cases.

The following assumptions are made

(i) That a zero-β asset exists.

(ii) That the LPM for the distributions of asset returns exists

These assumptions will be referred to as the existence assumptions.

Two fund separation is said to occur if given two distinct frontier portfolios, A and B, every frontier portfolio can be constructed through a linear combination of A and B. The degenerate case of this, when all assets have identical expected returns, produces *one fund separation*. In this case the efficient frontier is reduced to a single point – the minimum risk portfolio.

Definition (two fund separation): A portfolio exhibits two fund separation if there exists two funds, A and B, such that for any portfolio p there exists a scalar $\lambda \in \mathbb{R}$ such that $E[u(\lambda R_A + (1-\lambda)R_B)] \geq E\left[u\left(R_p\right)\right]$ for all $u \in U_1$ (concave utility).

Monetary separation occurs when a risk-free (zero-variance) asset is introduced with unrestricted borrowing and lending. In this case the investor chooses a linear combination of the riskless asset and the tangent portfolio.

A. Zero-β Separation in the ET framework

Consider a zero-β asset that has a return distribution with lower bound a and upper bound b. Let the stochastic return for this asset be R_z with probability distribution function $g(R_z)$. The expected return of this asset is thus:

$$E[R_z] = \int_a^b R_z g(R_z) dR_z \qquad [17]$$

I allow unrestricted borrowing and lending (purchase and shorting) of this asset.

Consider an arbitrary risky portfolio p with weights $\mathbf{w}$ such that $\mathbf{w}^{\mathrm{T}}\mathbf{1} = 1$ of m assets and associated stochastic return R_p. Now consider another portfolio q which is composed of an investment of $\mathbf{x}^{\mathrm{T}}\mathbf{1} \neq 0$ in portfolio p and $(1 - \mathbf{x}^{\mathrm{T}}\mathbf{1})$ in the zero-β asset.

The return for this portfolio q is:

$$R_q = \mathbf{x}^{\mathrm{T}}\mathbf{1}R_p + (1 - \mathbf{x}^{\mathrm{T}}\mathbf{1})R_z \qquad \text{so} \qquad R_p = \tfrac{1}{\mathbf{x}^{\mathrm{T}}\mathbf{1}}\left[R_q - (1 - \mathbf{x}^{\mathrm{T}}\mathbf{1})R_z\right] \qquad [18]$$

The normalized TPWL for R_q with a stochastic threshold $h = R_z$ is:

$$TPWL_n^{1/n}(R_q, R_z; a, b) = \left(\int_a^b \left[\int_{-\infty}^{R_z} (R_z - R_q)^n c(R_q) dR_q\right] dR_z \right)^{\frac{1}{n}} \qquad [19]$$

where $c(R_q)$ is the return distribution for portfolio q. I apply a transformation of variables to the inner integral.

$$R_q = \mathbf{x}^{\mathrm{T}}\mathbf{1}R_p + (1 - \mathbf{x}^{\mathrm{T}}\mathbf{1})R_z \qquad \text{so} \qquad dR_q = \mathbf{x}^{\mathrm{T}}\mathbf{1}dR_p$$

$$\text{and } c(R_q) = \mathcal{J}c(R_p, R_z) \text{ with Jacobian } \mathcal{J} = \tfrac{dR_p}{dR_q} = \tfrac{1}{\mathbf{x}^{\mathrm{T}}\mathbf{1}}$$

By definition R_z is independent of R_p and thus their joint distribution can be written as $c(R_p, R_z) = f(R_p)g(R_z)$ to form:

$$TPWL_n^{1/n}(R_q, R_z; a, b) = \left(\int_a^b \left[\int_{-\infty}^{R_z} (R_z - \mathbf{x}^{\mathrm{T}}\mathbf{1}R_p - (1 - \mathbf{x}^{\mathrm{T}}\mathbf{1})R_z)^n f(R_p)g(R_z)dR_p\right] dR_z\right)^{\frac{1}{n}} \quad [20]$$

$$= \left(\int_a^b \left[\int_{-\infty}^{R_z} \left(\mathbf{x}^{\mathrm{T}}\mathbf{1}(R_z - R_p)\right)^n f(R_p)dR_p\right] g(R_z)dR_z\right)^{\frac{1}{n}}$$

$$= \mathbf{x}^{\mathrm{T}}\mathbf{1}\left(\int_a^b \left[\int_{-\infty}^{R_z} (R_z - R_p)^n f(R_p)dR_p\right] g(R_z)dR_z\right)^{\frac{1}{n}}$$

$$= \mathbf{x}^{\mathrm{T}}\mathbf{1}TPWL_n^{1/n}(R_p, R_z; a, b) \quad [21]$$

This is an unusual form of two fund separation which I summarize in the following theorem.

> *Theorem 4 (Zero-β separation):* If a zero-β asset with a stochastic return exists then the portfolio composed of this asset and a risky portfolio is linearly separable in the ET framework with a stochastic threshold rate equal to the stochastic return of the zero-β asset.

The theorem can be expressed as a ratio:

$$\frac{E[R_q - R_z]}{TPWL_n^{1/n}(R_q, R_z; a, b)} = \frac{\mathrm{E}[\mathbf{x}^{\mathrm{T}}\mathbf{1}(R_p - R_z)]}{\mathbf{x}^{\mathrm{T}}\mathbf{1}TPWL_n^{1/n}(R_p, R_z; a, b)} = \frac{E[R_p - R_z]}{TPWL_n^{1/n}(R_p, R_z; a, b)} \quad [22]$$

B. Monetary Separation

Confusion regarding monetary separation within the EL-CAPM framework can be found within the literature. The determination of the threshold rates at which monetary separation obtains in the presence of a risk free asset has been one point in question. This issue is reviewed and clarified by Grootveld and Hallerbach (1999).

Bawa and Lindenberg (1977) considered monetary separation when a zero-variance asset is present and the threshold rate is the respective zero-variance rate. Although they showed mathematically that monetary separation obtains in the EL framework for all $n \geq 1$ they restricted their theorem to $n = 1,2$. Grootveld and Hallerbach (1999) showed that monetary separation obtains for all n. Bawa and Lindenberg (1977) claimed that monetary separation was the only form of separation to occur in the EL framework. Grootveld and Hallerbach (1999) showed this not to be the case.

Harlow and Rao (1989) showed that if asset distributions are assumed to belong to the two-parameter location-scale family then portfolio separation between the zero-variance asset and the market portfolio of risky assets obtains for an arbitrary threshold rate. Such distributions include the normal, logistic, student-t and uniform for example.

The zero-variance (two fund monetary separation) result of Bawa and Lindenberg obtains from Theorem 4 if the probability distribution function of the zero-β asset is restricted:

$$g(R_z) = \begin{cases} 1, & R_z = R_f \\ 0, & R_z \neq R_f \end{cases} \qquad [23]$$

where the return of the zero variance asset is represented by R_f. In this case the integral reduces to:

$$TPWL_n^{1/n}(R_q, R_f) = LPM_n^{1/n}(R_q, R_f) = \mathbf{x}^{\mathrm{T}}\mathbf{1}\left(\int_{-\infty}^{R_f}(R_f - R_p)^n f(R_p)dR_p\right)^{\frac{1}{n}} = \mathbf{x}^{\mathrm{T}}\mathbf{1}LPM_n^{1/n}(R_p, R_f) \qquad [24]$$

C. Mean Separation

Maintaining the existence assumptions, that the zero-β asset exists as do the LPMs for risky assets, I posit the following theorem on mean separation.

Theorem 5 (Mean Separation): If a zero-β asset with a stochastic return exists then the portfolio composed of this asset $(1-\mathbf{x}^{\mathrm{T}}\mathbf{1})$ and a risky portfolio $(\mathbf{x}^{\mathrm{T}}\mathbf{1})$ is linearly separable in the ET framework with a threshold rate equal the sum of $\mathbf{x}^{\mathrm{T}}\mathbf{1}$ times the expected return of the risky portfolio and $(1-\mathbf{x}^{\mathrm{T}}\mathbf{1})$ times the stochastic return of the zero-β asset.

$$TPWL_n^{1/n}\left(R_q, \mathbf{x}^{\mathrm{T}}\mathbf{1}E\left[R_p\right] + (1-\ \mathbf{x}^{\mathrm{T}}\mathbf{1})R_z; a, b\right) = \mathbf{x}^{\mathrm{T}}\mathbf{1}TPWL_n^{1/n}\left(R_p, E\left[R_p\right]; a, b\right) \quad [25]$$

where $R_q = \mathbf{x}^{\mathrm{T}}\mathbf{1}R_p + (1-\ \mathbf{x}^{\mathrm{T}}\mathbf{1})R_z$

The proof of this theorem can be found in Appendix A.

Within the EL literature additional confusion exists regarding 'mean separation'. This form of separation has been considered in the presence of a zero-variance asset. Grootveld and Hallerbach (1999) noted that Lee and Rao (1988) mistakenly claimed that for $n = 1{,}2$ linear separation occurs when the threshold rate equals that of the return of the tangency portfolio.

Grootveld and Hallerbach (1999) showed that linear separation occurs when the threshold return is the expected return of a convex combination of the zero-variance asset and the tangent portfolio. They distinguished between three cases of threshold return; fixed, deterministic yet variable, and stochastic[8]

[8] Recently Brogan and Stidham Jr (2005) stated that the Grootveld and Hallerbach (1999) condition was incorrect because the threshold 'cannot be fixed but must remain the mean of the (mixed) portfolio under consideration'. This view is mistaken as Grootveld and Hallerbach do not fix the threshold beyond saying that it is the expected return of the portfolio q (the mixed portfolio). Indeed Grootveld and Hallerbach say that this condition for linear separation is 'problematic in the sense that the target (threshold) rate is not constant and depends on the optimal investment portfolio'. This mistake may have arisen from a misunderstanding on means and

If the probability distribution function of the zero-β asset is restricted as in equation [23] the EL mean separation result of Grootveld and Hallerbach (1999) is obtained:

$$LPM_n^{1/n}\left(R_q, \mathbf{x}^{\mathrm{T}}\mathbf{1}E\left[R_p\right] + (1 - \mathbf{x}^{\mathrm{T}}\mathbf{1})R_f\right) = \mathbf{x}^{\mathrm{T}}\mathbf{1}LPM_n^{1/n}\left(R_p, E\left[R_p\right]\right) \qquad [26]$$

where $R_q = \mathbf{x}^{\mathrm{T}}\mathbf{1}R_p + (1 - \mathbf{x}^{\mathrm{T}}\mathbf{1})R_f$ *and* $E\left[R_q\right] = \mathbf{x}^{\mathrm{T}}\mathbf{1}E\left[R_p\right] + (1 - \mathbf{x}^{\mathrm{T}}\mathbf{1})R_f$

expected values. Brogan and Stidham Jr (2008) prove a theorem which, apart from terminology and nomenclature, appears to be identical to that of Grootveld and Hallerbach (1999).

V. The ET-CAPM

Building on the zero-β portfolio separation result of Theorem 4, I now present a new partial equilibrium asset pricing model consistent with the notion of zero-β assets. I derive the 'ET-CAPM' by following the approach used by Sharpe (1964). This same approach was adapted by Hogan and Warren (1974) and Bawa and Lindenberg (1977) in their derivations of the EL-CAPM.

A. The Derivation

Again the existence assumptions are made, that the zero-β asset exists as do the LPMs for risky assets. Let h be the stochastic return R_z on a zero-β asset, with lower and upper bounds $[a, b]$. Consider a risky portfolio p constituted only of risky assets, with associated stochastic return R_p and weights $\mathbf{w}$ such that $\mathbf{w}^{\mathrm{T}}\mathbf{1} = 1$ of m assets. Now assume this risky portfolio lies on the ET efficient frontier (for portfolios consisting of risky assets only) and further that this portfolio is the tangent portfolio as seen in Figure 1 below. Now consider another portfolio q which is composed of an investment of $\mathbf{x}^{\mathrm{T}}\mathbf{1}$ in some risky asset j and $(1 - \mathbf{x}^{\mathrm{T}}\mathbf{1})$ in the tangent portfolio p.

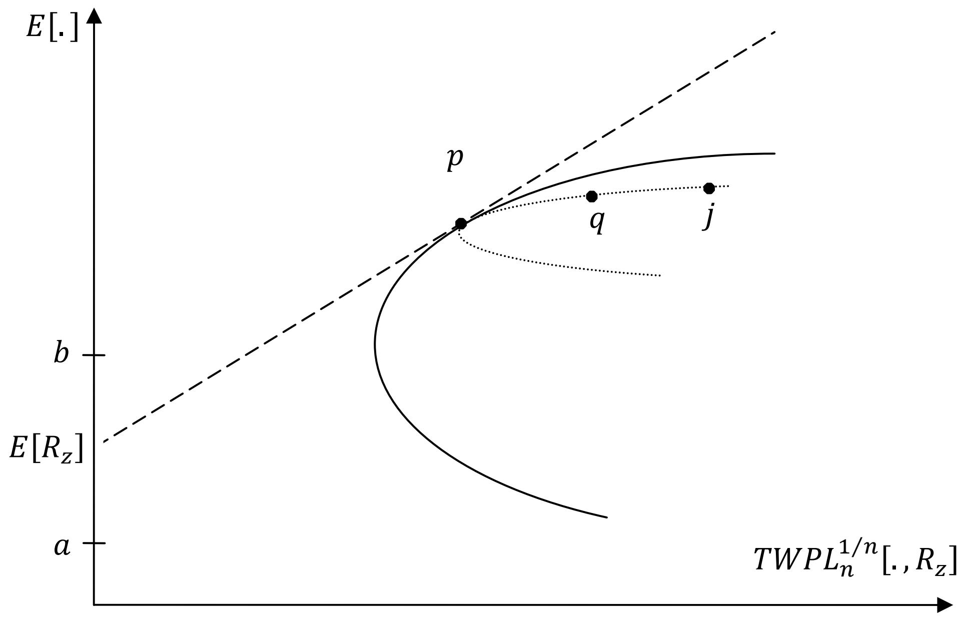
The return for this portfolio q is:

$$R_q = \mathbf{x}^{\mathrm{T}}\mathbf{1}R_j + (1 - \mathbf{x}^{\mathrm{T}}\mathbf{1})R_p \qquad \text{and} \qquad E[R_q] = \mathbf{x}^{\mathrm{T}}\mathbf{1}E[R_j] + (1 - \mathbf{x}^{\mathrm{T}}\mathbf{1})E[R_p] \qquad [27]$$

FIGURE 1

Efficient Frontier in ET Space

Figure 1 presents the efficient frontier in ET space. a and b are the lower and upper bounds for the return on the zero-β asset, R_z. The tangent portfolio p is constituted only of risky assets and portfolio q is composed of an investment of $\mathbf{x}^{\mathrm{T}}\mathbf{1}$ in some risky asset j and $(1 - \mathbf{x}^{\mathrm{T}}\mathbf{1})$ in the tangent portfolio p.



By equating the gradient of the curve pj with that of the tangent line and much algebra I arrive at this relationship involving moments and partial moments:

$$E[R_j - R_z] = {}_{n}^{ET}\beta_j E[R_p - R_z] \qquad n \geq 1 \tag{28a}$$

$${}_{n}^{ET}\beta_j = \frac{CTPWL_n[R_j, R_p, R_z]}{TPWL_n[R_p, R_z]} \tag{28b}$$

$$CTPWL_n[R_j, R_p, R_z] = E\left[(R_z - R_p)^{+(n-1)}(R_z - R_j)\right] \tag{28c}$$

$$TPWL_n[R_p, R_z] = E\left[(R_z - R_p)^{+n}\right] \tag{28d}$$

The full derivation can be found in Appendix B. In order to emphasize the role of the probability distribution function for R_z I rewrite this as:

$$\int_a^b \left[\int_{-\infty}^{\infty} (R_j - R_z) f(R_j) dR_p\right] g(R_z) dR_z = {}^{ET}_{n}\beta_j \int_a^b \left[\int_{-\infty}^{\infty} (R_p - R_z)\, f(R_p) dR_p\right] g(R_z) dR_z \qquad [29a]$$

$${}^{ET}_{n}\beta_j(R_p; R_z, R_j) = \frac{\int_a^b \int_{-\infty}^{+\infty} \int_{-\infty}^{R_z} (R_z - R_p)^{n-1} (R_z - R_j)\, e(R_j, R_p) dR_p dR_j g(R_z) dR_z}{\int_a^b \int_{-\infty}^{R_z} (R_z - R_p)^n\, f(R_p) dR_p\, g(R_z) dR_z} \qquad [29b]$$

with $e(R_j, R_p)$ being the joint distribution of R_j and R_p.

One way to complete the model is to incorporate a significant economic assumption; that the market portfolio is the tangent portfolio. There is a significant amount of literature covering mean-variance efficiency however here I assume ET-efficiency. The model then becomes:

$$E[R_j - R_z] = {}^{ET}_{n}\beta_j E[R_m - R_z] \qquad [30a]$$

$${}^{ET}_{n}\beta_j = \frac{CTPWL_n[R_j, R_m, R_z]}{TPWL_n[R_m, R_z]} \qquad [30b]$$

B. Model features

The key assumptions in this model are that (i) investors prefer portfolios with higher ET efficiency and (ii) that the market portfolio will be the tangent portfolio. Whilst the market portfolio is assumed ET efficient, it will generally not be EL efficient.

The ET Beta is not a weighted sum of LPM Betas. The numerator of the ET Beta is a weighted sum of Co-LPMs and the denominator is a weighted sum of LPMs. The LPMs and Co-LPMs differ in their threshold. The weights are the probability of the threshold, that is, the probability of each zero-β return outcome.

The probability distribution function of the zero-β asset must be determined or assumed in applications of this model. Options markets on short term debt could be used to infer this distribution[9].

Theorem 1 contained in Bawa and Lindenberg (1977) states that distribution F_A dominates asset F_B if and only if the LPM of F_A is equal to or less than that of F_B for all values of the threshold. The EL-CAPM of Bawa and Lindenberg (1977), whilst motivated by this theorem, only accommodates a single threshold value on the return domain. Thus the EL-CAPM is based on a necessary but not sufficient condition for stochastic dominance within a class of utility. Similarly, the ET-CAPM motivated by Theorem 3 accommodates stochastic dominance over a particular subinterval of the return domain under an assumed form for utility; it accommodates partial stochastic dominance. Therefore the ET-CAPM has a similar link with utility theory as that of the EL-CAPM. The ET-CAPM is also based on necessary but not sufficient conditions for stochastic dominance within a class of utility.

Many earlier pricing models can be seen as special cases of the ET-CAPM. These models appear when restrictions are placed on (i) the upper and lower bounds for the return distribution of the zero-β asset and (ii) the form of the return distribution for the risky assets. The following table indicates how this model encapsulates many of the early asset pricing models.

[9] For a review of these techniques see Bliss and Panigirtzoglou (2002).

TABLE 1

Earlier Asset Pricing Models as Special Cases of the ET-CAPM

TPWL Degree	$g(R_z)$ Restrictions	Risky Asset Distributional Requirements	Equivalent Model	Notes
2	$g(R_z) = \begin{cases} 1, & R_z = R_f \\ 0, & R_z \neq R_f \end{cases}$	Nil	Hogan & Warren 1974	This model uses a zero variance risk free rate
1,2	$g(R_z) = \begin{cases} 1, & R_z = R_f \\ 0, & R_z \neq R_f \end{cases}$	Nil	Bawa & Lindenberg 1977	This model uses a zero variance risk free rate
2	$g(R_z) = \begin{cases} 1, & R_z = R_f \\ 0, & R_z \neq R_f \end{cases}$	Elliptically Symmetric Distributions	Sharpe / Lintner 1964/1965	This model uses a zero variance risk free rate
2	The zero-β asset is by definition uncorrelated.	Elliptically Symmetric Distributions	Black 1972	This model uses a zero-β asset. Black's CAPM does not require elliptically symmetric distributions. It is necessary here in order for ${}^{ET}_{n}\beta_j$ to be equivalent to the Black Beta.

VI. The Pricing Kernel

From consideration of the representative investor's Lagrangian the utility function for the ET-CAPM is

$$U(R) = \begin{cases} U^{+}(R) = \theta_1 + \theta_2 R_m^e & for\ R_m^e \geq 0 \\ U^{-}(R) = \theta_1 + \theta_2 R_m^e - \theta_3 (-R_m^e)^n & for\ R_m^e < 0 \end{cases} \quad [31]$$

Thus the pricing kernel for the ET-CAPM is

$$M^{ET} = 1 + \theta (-R_m^e)^{+n-1} \quad [32]$$

Where

$$\theta = n\frac{\theta_3}{\theta_2} \quad [33]$$

Pricing kernels that can adopt negative values pose two clear difficulties; the suggestion of assets with negative expected returns and arbitrage opportunities. The ET-CAPM pricing kernel is strictly positive, non-linear, and convex function of the market's excess return over the zero-beta asset. Additionally, the pricing kernel can be equated with the marginal rate of substitution.

To derive the ET-CAPM from this kernel I start with the identity

$$E[R_j^e] = \frac{Cov[M, R_j^e]}{Cov[M, R_m^e]} E[R_m^e] \quad [34]$$

substituting M^{ET} for M and simplifying gives

$$E[R_j^e] = \frac{Cov[(-R_m^e)^{+n-1}, R_j^e]}{Cov[(-R_m^e)^{+n-1}, R_m^e]} E[R_m^e] \quad [35]$$

Prima facie this appears to be inconsistent with the ET-CAPM which involves the weighted sums of co-lower partial moments and partial moments; co-lower partial moments are not equivalent to covariances. Furthermore lower partial moments do not possess a decomposition rule. However the ET-CAPM involves the ratio of the sums co-lower partial moments. An important lemma on such ratios follows and is proven in Appendix C. This result seems absent from the literature which has focused on the properties (the lack of any) of the lower partial moment alone.

Lemma: If the following expression holds between three random variables X, Y and Z

$$\frac{Cov\left(X^{+^{n-1}}, Y\right)}{Cov\left(X^{+^{n-1}}, Z\right)} = \frac{E(Y)}{E(Z)} \qquad [36]$$

then

$$\frac{CLPM_n(X, Y)}{CLPM_n(X, Z)} = \frac{Cov\left(X^{+^{n-1}}, Y\right)}{Cov\left(X^{+^{n-1}}, Z\right)} = \frac{E(Y)}{E(Z)} \qquad [37]$$

This can be readily extended for the use within the ET framework. The theorem below is also proven in Appendix C.

Theorem: If the following expression holds between three random variables X, Y and Z and a fourth uncorrelated variable ε

$$\frac{Cov\left((X-\varepsilon)^{+^{n-1}}, Y-\varepsilon\right)}{Cov\left((X-\varepsilon)^{+^{n-1}}, Z-\varepsilon\right)} = \frac{E(Y-\varepsilon)}{E(Z-\varepsilon)} \qquad [38]$$

then

$$\frac{CTPWL_n(X, Y, \varepsilon)}{CTPWL_n(X, Z, \varepsilon)} = \frac{Cov\left((X-\varepsilon)^{+^{n-1}}, Y-\varepsilon\right)}{Cov\left((X-\varepsilon)^{+^{n-1}}, Z-\varepsilon\right)} = \frac{E(Y-\varepsilon)}{E(Z-\varepsilon)} \qquad [39]$$

By substituting $-R_z$ for ε, $-R_j$ for Y, and $-R_m$ for both X and Z the ET-CAPM follows immediately. Similarly, the pricing kernel for the EL-CAPM of (Hogan and Warren (1974) & Bawa and Lindenberg (1977)) is

$$M^{EL} = 1 + \theta\left(R_f - R_m\right)^+ \qquad [40]$$

The EL-CAPM can be derived by substituting setting $n = 2$, $Y = R_f - R_j$ and $X = Z = R_f - R_m$ into the Lemma (equation 37).

VII. Divergent Borrowing and Lending Facilities

In this section we allow for distinct borrowing and lending using two zero-β assets. The first of these assets acts a lending facility that has a return distribution with lower and upper bounds a_L and b_L. The second of these assets acts a borrowing facility that has a return distribution with lower and upper bounds a_B and b_B.

Let the stochastic return for these assets be R_z^L and R_z^B with expected returns (for all investors)

$$E[R_z^L] = \int_{a_L}^{b_L} R_z^L e(R_z^L) dR_z^L \qquad [41a]$$

$$E[R_z^B] = \int_{a_B}^{b_B} R_z^B h(R_z^B) dR_z^B \qquad [41b]$$

where $e(R_z^L)$ and $h(R_z^B)$ are the probability distribution functions for R_z^L and R_z^B respectively.

A necessary restriction for these assets is $E[R_z^B] \geq E[R_z^L]$. Any quantity lent must be strictly non-negative and the quantity borrowed must be non-positive. The joint distribution of returns for these assets will be denoted by $g(R_z^L, R_z^B)$.

Consider an arbitrary risky portfolio p with stochastic return R_p constituted of l assets with weights $\mathbf{w}$ such that $\mathbf{w}^{\mathrm{T}}\mathbf{1} = 1$ Now consider another portfolio q which is composed of an investment of $\mathbf{x}^{\mathrm{T}}\mathbf{1} \neq 0$ in portfolio p and $\delta(1 - \mathbf{x}^{\mathrm{T}}\mathbf{1})$ in the lending facility and $(1-\delta)(1 - \mathbf{x}^{\mathrm{T}}\mathbf{1})$ in the borrowing facility. The return for this portfolio q is:

$$R_q = \mathbf{x}^{\mathrm{T}}\mathbf{1}R_p + \delta(1 - \mathbf{x}^{\mathrm{T}}\mathbf{1})R_z^L - (1-\delta)(1 - \mathbf{x}^{\mathrm{T}}\mathbf{1})R_z^B$$

$$R_p = \frac{1}{\mathbf{x}^{\mathrm{T}}\mathbf{1}}\left[R_q - (1 - \mathbf{x}^{\mathrm{T}}\mathbf{1})\delta(1 - \mathbf{x}^{\mathrm{T}}\mathbf{1})R_z^L + (1-\delta)(1 - \mathbf{x}^{\mathrm{T}}\mathbf{1})R_z^B\right] \qquad [42]$$

I write the normalized TPWL with a stochastic threshold $h = \delta R_z^L - (1-\delta)R_z^B$ as:

$$TPWL_n^{1/n}\left(R_q, \delta R_z^L - (1-\delta)R_z^B; a_L, b_L, a_B, b_B\right)$$

$$= \left(\int_{a_L}^{b_L}\int_{a_B}^{b_B}\left[\int_{-\infty}^{\delta R_z^L + (1-\delta)R_z^B}\left(\delta R_z^L - (1-\delta)R_z^B - R_q\right)^n c\left(R_q\right)dR_q\right]dR_z^L\, dR_z^B\right)^{\frac{1}{n}} \qquad [43]$$

I apply a transformation of variables to the inner integral.

$$R_q = \mathbf{x}^{\mathrm{T}}\mathbf{1}R_p + \delta(1-\mathbf{x}^{\mathrm{T}}\mathbf{1})R_z^L - (1-\delta)(1-\mathbf{x}^{\mathrm{T}}\mathbf{1})R_z^B \quad \text{so} \quad dR_q = \mathbf{x}^{\mathrm{T}}\mathbf{1}dR_p$$

$$\text{and } c\left(R_q\right) = \mathcal{J}c\left(R_p, R_z^L, R_z^B\right) \text{ with Jacobian } \mathcal{J} = \frac{dR_p}{dR_q} = \frac{1}{\mathbf{x}^{\mathrm{T}}\mathbf{1}}$$

By definition R_z^L and R_z^B are each independent of R_p and thus their joint distribution can be written as $c(R_p, R_z^L, R_z^B) = f(R_p)g(R_z^L, R_z^B)$ to form:

$$TPWL_n^{1/n}\left(R_q, \delta R_z^L - (1-\delta)R_z^B; a_L, b_L, a_B, b_B\right)$$

$$= \left(\int_{a_L}^{b_L}\int_{a_B}^{b_B}\left[\int_{-\infty}^{\delta R_z^L + (1-\delta)R_z^B}\begin{pmatrix}\delta\mathbf{x}^{\mathrm{T}}\mathbf{1}R_z^L - (1-\delta)\mathbf{x}^{\mathrm{T}}\mathbf{1}R_z^B \\ -\mathbf{x}^{\mathrm{T}}\mathbf{1}R_p\end{pmatrix}^n f\left(R_p\right)g(R_z^L, R_z^B)\, dR_p\right]dR_z^L\, dR_z^B\right)^{\frac{1}{n}}$$

$$= \mathbf{x}^{\mathrm{T}}\mathbf{1}\left(\int_{a_L}^{b_L}\int_{a_B}^{b_B}\left[\int_{-\infty}^{\delta R_z^L + (1-\delta)R_z^B}\left(\delta R_z^L - (1-\delta)R_z^B - R_p\right)^n f\left(R_p\right)g(R_z^L, R_z^B)\, dR_p\right]dR_z^L\, dR_z^B\right)^{\frac{1}{n}}$$

$$= \mathbf{x}^{\mathrm{T}}\mathbf{1}TPWL_n^{1/n}\left(R_p, \delta R_z^L - (1-\delta)R_z^B; a_L, b_L, a_B, b_B\right) \qquad [44]$$

This form of fund separation is summarized in the following theorem.

Theorem 6 (Separation with Divergent Borrowing and Lending Facilities): Consider two zero-β assets such that $E[R_z^B] \geq E[R_z^L]$. The portfolio composed of these assets and the risky portfolio with weights $\delta\left(1-\mathbf{x}^{\mathrm{T}}\mathbf{1}\right)$, $-(1-\delta)\left(1-\mathbf{x}^{\mathrm{T}}\mathbf{1}\right)$, and $\mathbf{x}^{\mathrm{T}}\mathbf{1}$, is linearly separable with a stochastic threshold rate equal to the stochastic return, $h = \delta R_z^L - (1-\delta)R_z^B$.

It is clear here that a rational investor will have $\delta = 1$ or $\delta = 0$; they will avoid borrowing only to lend with a lower expected return. In the special case where $R_z^B = R_z^L + \gamma$, with γ a constant, the stochastic return simplifies to $h = (2\delta - 1)R_z^L + \gamma(\delta - 1)$.

The ET frontier therefore will be curvilinear, see Figure 2. All investors would hold either

(i) the single risky portfolio (equivalent to p^L) in conjunction with lending

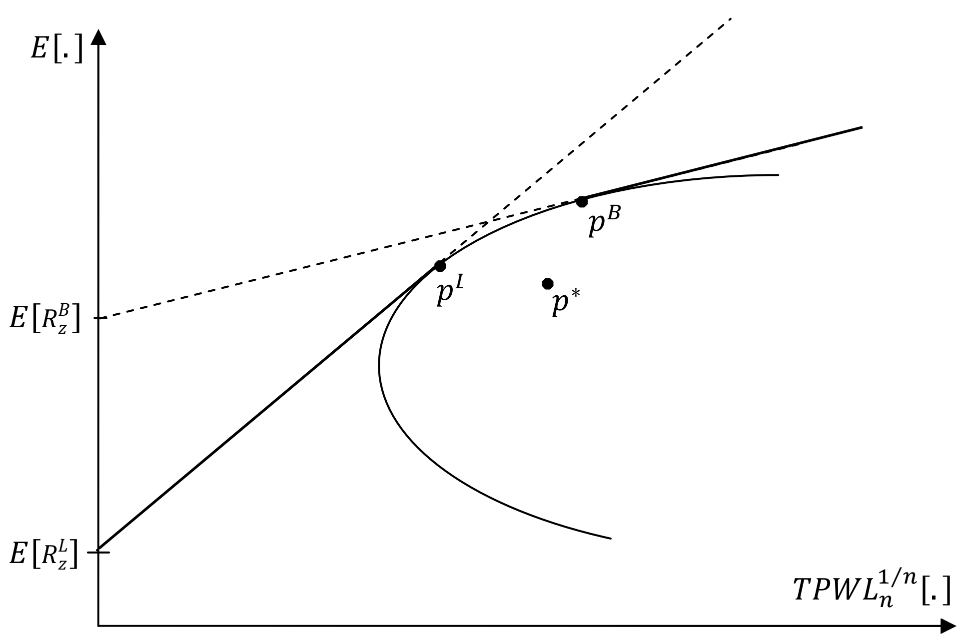
(ii) the single risky portfolio (equivalent to p^B) in conjunction with borrowing

(iii) a single risky portfolio of the partial frontier $p^L - p^B$.

FIGURE 2

Efficient Frontier in ET Space

Figure 2 presents the efficient frontier in ET space with divergent lending and borrowing facilities. The frontier is curvilinear and consists three parts; the line leading to p^L associated with net lenders, the partial frontier, $p^L - p^B$associated with neither borrowing nor lending, and lastly the line extending away from p^B associated with net borrowers. The portfolio p^* represents a linear combination of p^L and p^B, which will not in general rest on the frontier.



The equivalent observation in the EV framework was attributed by Brennan (1971) to a working paper[10] by Blume and Friend (1973). All investors in the EV-framework would hold either

(i) the single risky portfolio (equivalent to p^L) in conjunction with lending

(ii) the single risky portfolio (equivalent to p^B) in conjunction with borrowing

(iii) a convex combination of two risky portfolios, (equivalent to, $\lambda_L p^L + \lambda_B p^B$ with $\lambda_L + \lambda_B=1$).

The fact that two distinct EV-frontier portfolios will span the entire EV-frontier prima facie suggests a linear pricing relation involving covariance and a shifted intercept should be possible, however this is only the case if significant constraints are put on utility functions; that they are parabolic in mean-standard deviation space (Brennan (1971)). The result cannot be derived directly from quadratic utility (Chua (1975)). This implicit assumption allowed Brennan to derive a linear pricing model whose only difference with the conventional form is a shifting of the tangent's intercept.

Such a spanning result (and in fact the entire hyperbolic geometry) is not present in the ET (or EL) framework in general. Hence a simple general linear pricing relation from Theorem 4 incorporating the market portfolio cannot be derived for divergent borrowing and lending facilities in this (or the EL) framework.

[10] The material cited by Brennan in the working paper of Blume and Friend (1973) appears to have been omitted when published in a journal.

VIII. An Illustration of the Model

The data for this demonstration spans July-1954 to December-2008 and consists of the 30 value weighted Fama-French industry portfolios[11], the value weighted CRSP index, and the monthly return generated by investing at the daily effective Federal Funds rate.

The monthly return generated by investing at the daily effective Federal Funds rate is used as the zero-β asset, assumed uniformly distributed[12]. I formulate the problem in terms of excess returns above that of zero-β asset. The value of n is required to be greater than or equal to 1. Here it is set to 2, consistent with many of the early pricing models found in Table 1. As an additional illustration, the case of $n = 5$ can be found in Appendix D.

The lower and upper bounds for the zero-β asset (around its expected value) are set to -1.5% and $+1.5\%$ per annum respectively[13]. This crudely represents the empirical history of the zero-β asset[14]. Sources of variability include supply-demand effects, monetary policy, regulatory or interbank credit changes. The zero-β return domain is sliced into 150 intervals of 2 basis points for the purposes of TPWL calculation.

The ET efficient frontier, the upper and lower bounds of the zero-β asset, the locations of the 30 industry portfolios, the market portfolio and the tangent portfolio are all depicted in Figure 3. The definitions of the 30 industry portfolios can be found in Appendix E.

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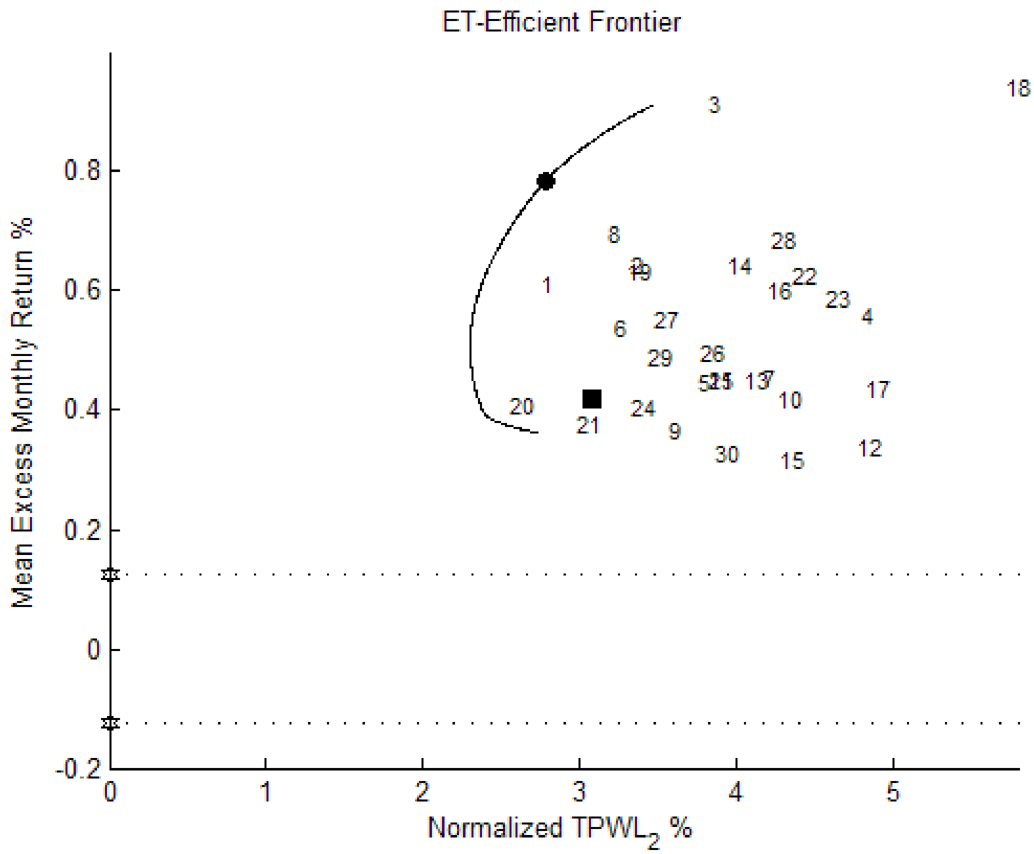
[11] Provided by Fama and French (available from Kenneth French's web site at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/).
[12] Any bounded distribution for which the lower partial moment exists could be used. For simplicity I have chosen the uniform distribution.
[13] Extending these bounds altered the results little.
[14] The daily effective US Federal Funds rate is the volume-weighted average of rates on trades arranged by major brokers (banks and savings & loans associations). Although the effective Federal Funds rate has historically been close to the target rate set by the Federal Reserve recently (September to November 2008) it has differed from the target rate by as much as 133 basis points. Other large deviations can be found in times of market stress; August 2007 (75 bps) and September 2001 (180 bps).

FIGURE 3

Efficient Frontier in ET Space

Figure 3 presents the efficient frontier in ET space. The marks on the return axis indicate the upper and lower pounds of the zero-β asset. The disc on the frontier indicates the tangent portfolio. The square indicates the market portfolio. The industry portfolios are numbered 1 through 30. The definitions can be found in Appendix E.



The distance between the market portfolio and the tangent portfolio must be placed in context; this frontier is for a particular selection of n, the investable assets consist only of US equities, and the return distribution for the zero-β asset is rudimentary. The betas against the tangent portfolio are plotted in Figure 4. The betas against the CRSP market index are plotted in 5.

FIGURE 4

Industry Portfolio (Tangent) Betas

Figure 4 presents the tangent ET Betas against the mean monthly excess return for each of the industry portfolios. The disc indicates the location of the tangent portfolio. The industry portfolios are numbered 1 through 30. The definitions can be found in Appendix E.

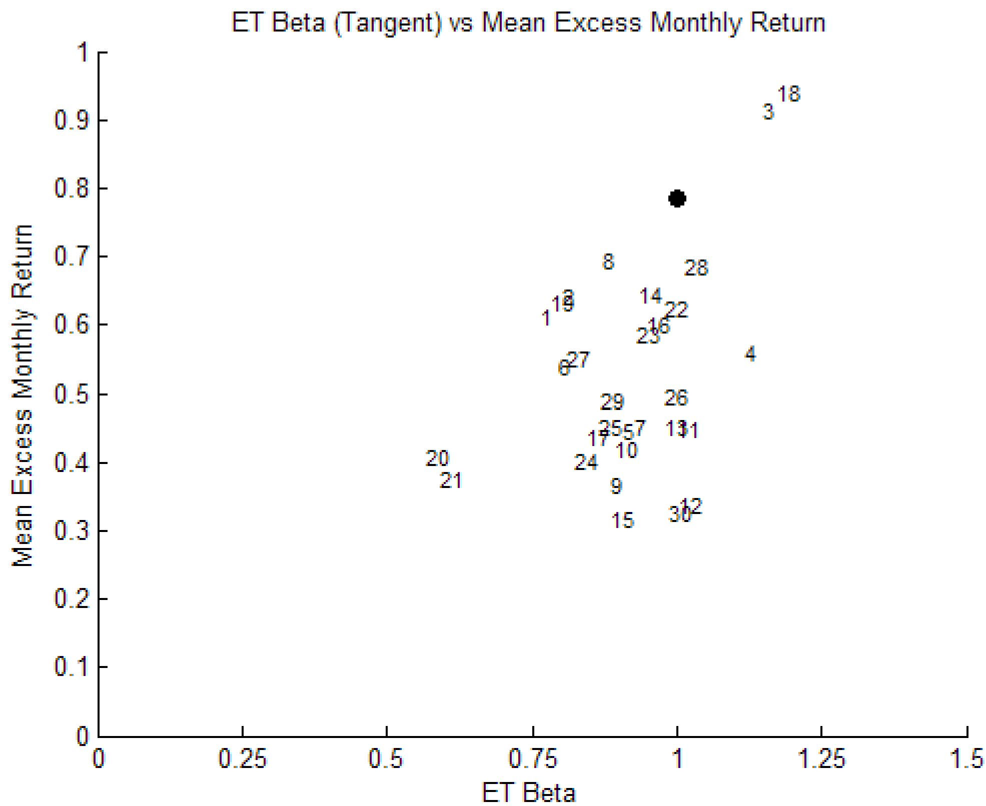
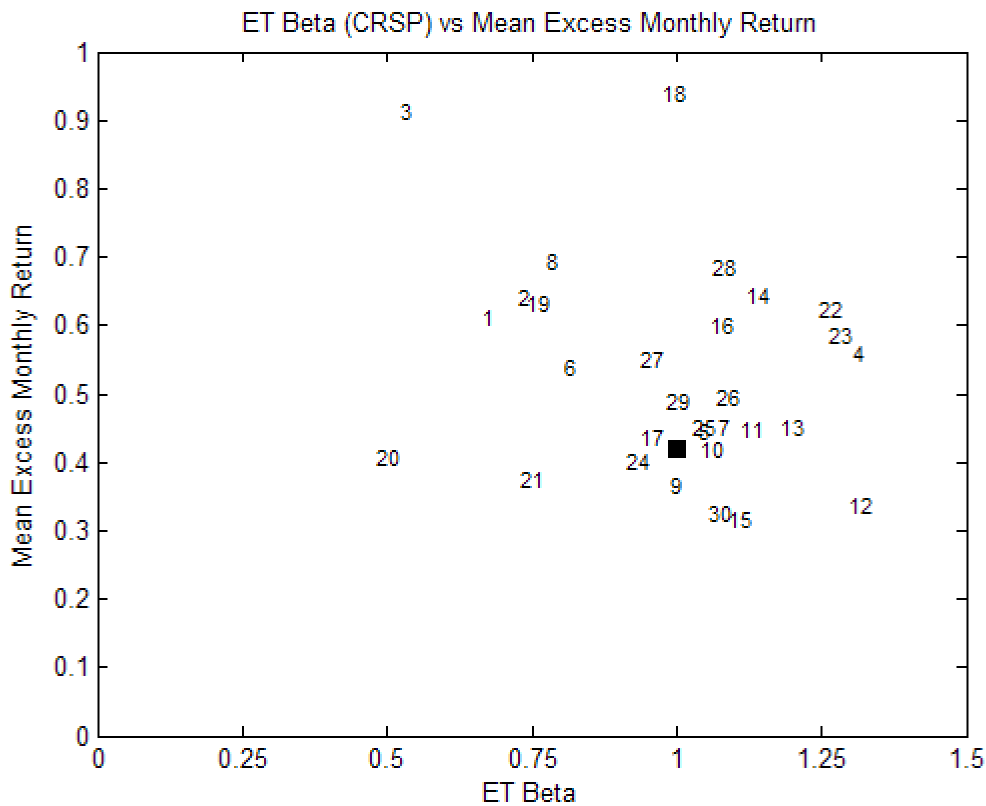


FIGURE 5

Industry Portfolio (CRSP Market) Betas

Figure 5 presents the CRSP Market ET Betas against the mean monthly excess return for each of the industry portfolios. The square indicates the CRSP market index. The industry portfolios are numbered 1 through 30. The definitions can be found in Appendix D.



IX. Empirical Testing

In this section the ET-CAPM ($n = 2$) is tested using Hansen's generalized method of moments (GMM). Other values on n are tested and a Bonferroni correction applied to the collection of results. The data spans July-1954 to December-2008 and consists of the 30 value weighted Fama-French industry portfolios, the value weighted CRSP index, and the monthly return generated by investing at the daily effective Federal Funds rate[15]. The process $R_{j,t}^{z} = \alpha_j + {}_{n}^{ET}\beta_j R_{m,t}^{z} + \varepsilon_{j,t}$ cannot be used to test the EL-CAPM (or ET-CAPM) as it does not distinguish between asymmetric and traditional forms of CAPM (see Appendix in Chow and Denning (1994)). Historically the EL-CAPM has been tested by with an asymmetric return model (Bawa, Brown and Klein (1981), Harlow and Rao (1989)).

In this paper a GMM test is applied directly. By subtracting the Effective Federal Funds rate from the industry portfolios $\left(R_j^z = R_j - R_z\right)$ and the index $(R_m^z = R_m - R_z)$ the model can be rewritten as

$$E\left[R_j^z\right] - {}_{n}^{ET}\beta_j E[R_m^z] = 0 \quad [45]$$

From the definition of ${}_{n}^{ET}\beta_j$

$${}_{n}^{ET}\beta_j E\left[(-R_m^z)^{+n}\right] - E\left[-R_j^z(-R_m^z)^{+(n-1)}\right] = 0 \quad [46]$$

These equations form the basis for the moment restrictions $g_T(\theta)$:

[15] The monthly return generated by investing at the daily effective Federal Funds rate is a natural choice for the zero-β asset within the ET-framework. It is stochastic, uncorrelated with risky asset returns, and the return distribution has a small left tail with a lower bound of zero. The LPM of this return about an appropriately selected threshold (zero would be a conservative selection) is zero. A collection of bonds is also stochastic and uncorrelated with risky assets. However, like most risky assets, it has a lower bound on returns of -100%.

$$g_T(\theta) = \frac{1}{T}\sum_{t=1}^{T}\begin{bmatrix} \alpha_j - R_{j,t}^z + {}_{n}^{ET}\beta_j\, R_{m,t}^z \\ {}_{n}^{ET}\beta_j(-R_m^z)^{+n} + R_{j,t}^z\left(-R_{m,t}^z\right)^{+(n-1)} \end{bmatrix} \quad for\ \ j = 1\ to\ 30 \qquad [47]$$

where T is the number of observations and $\theta = (\alpha_1, \cdots, \alpha_{30}, {}_{n}^{ET}\beta_1, \cdots, {}_{n}^{ET}\beta_{30})$ are the parameters to be estimated for a given n. With $n = 2$ this becomes

$$g_T(\theta) = \frac{1}{T}\sum_{t=1}^{T}\begin{bmatrix} \alpha_j - R_{j,t}^z + {}_{n}^{ET}\beta_j\, R_{m,t}^z \\ {}_{n}^{ET}\beta_j(-R_m^z)^{+2} + R_{j,t}^z\left(-R_{m,t}^z\right)^{+} \end{bmatrix} \quad for\ \ j = 1\ to\ 30 \qquad [48]$$

The null hypothesis is

$$H_0\colon \bar{\alpha} = \mathbf{0} \qquad [49]$$

The moment restrictions constitute 60 equations with 30 unknowns; the 30 industry betas. The Newey West difference test for over-identifying restrictions is conducted by setting $\bar{\alpha} = \mathbf{0}$ and calculating the difference statistic.

$$TJ_{T(restricted)} - TJ_{T(unrestricted)} \sim \chi^2_{(number\ of\ restrictions)} \qquad [50]$$

The results contained within Table 2 indicate that the model cannot be rejected at the 5% significance level (p-value of 0.07).

TABLE 2

ET-CAPM - GMM Estimation of Parameters in Unrestricted Model and Test of Overindentifying Restrictions

Table 2 displays the parameter estimates for each of the 30 Fama-French industry portfolios from the unrestricted and restricted versions of the ET-CAPM. The J statistic and p-value (0.0704) for the Null implies that the model cannot be rejected at the 5% level. The data spans July-1954 to December-2008.

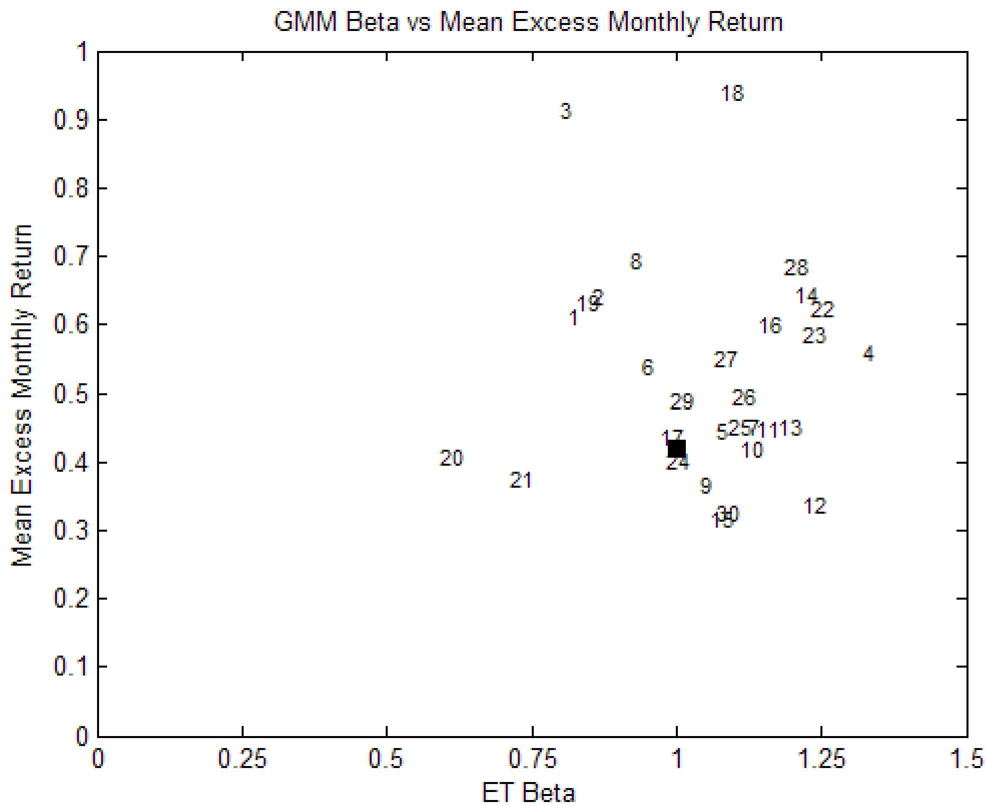
	Unrestricted Model						Restricted Model		
	Alphas			Betas			Betas		
Industry Portfolio	**Estimate**	**Standard Error**	**T-Ratio**	**Estimate**	**Standard Error**	**T-Ratio**	**Estimate**	**Standard Error**	**T-Ratio**
Food	0.003	0.001	-2.216	0.677	0.059	11.402	0.823	0.037	22.555
Beer & Liquor	0.003	0.002	-1.851	0.739	0.076	9.737	0.865	0.048	17.840
Tobacco Products	0.007	0.003	-2.673	0.535	0.092	5.803	0.810	0.059	13.801
Recreation	0.000	0.002	-0.035	1.314	0.068	19.466	1.332	0.046	28.981
Publishing	0.000	0.002	-0.023	1.050	0.063	16.771	1.080	0.040	27.010
Consumer Goods	0.002	0.001	-1.323	0.818	0.063	12.974	0.952	0.042	22.598
Apparel	0.000	0.002	0.026	1.085	0.076	14.249	1.135	0.051	22.109
Healthcare	0.004	0.001	-2.466	0.788	0.059	13.392	0.933	0.041	22.970
Chemicals	-0.001	0.001	0.402	1.000	0.057	17.453	1.054	0.035	30.245
Textiles	0.000	0.002	0.142	1.062	0.084	12.685	1.131	0.054	21.039
Construction	0.000	0.001	0.207	1.131	0.050	22.815	1.161	0.028	41.898
Steel	-0.002	0.002	1.073	1.320	0.083	15.982	1.238	0.054	22.737
Fabricated Products	-0.001	0.001	0.379	1.201	0.052	23.057	1.196	0.033	36.693
Electrical	0.002	0.001	-1.212	1.143	0.046	24.824	1.226	0.036	34.136
Automobiles	-0.002	0.002	0.800	1.111	0.074	14.935	1.081	0.046	23.363
Transport Equipment	0.001	0.002	-0.729	1.080	0.069	15.681	1.162	0.045	25.802
Mining	0.000	0.003	-0.117	0.959	0.106	9.058	0.994	0.068	14.710
Coal	0.005	0.004	-1.279	0.996	0.167	5.971	1.098	0.090	12.242
Oil	0.003	0.002	-1.766	0.762	0.057	13.444	0.847	0.042	19.989
Utilities	0.002	0.001	-1.336	0.504	0.063	8.026	0.612	0.041	14.767
Communication	0.001	0.001	-0.407	0.751	0.051	14.712	0.734	0.036	20.305
Services	0.001	0.002	-0.571	1.266	0.053	23.703	1.253	0.035	35.911
Business Equipment	0.000	0.002	-0.255	1.284	0.071	18.091	1.238	0.045	27.405
Paper	0.000	0.001	-0.064	0.935	0.047	19.841	1.006	0.029	34.163
Transportation	0.000	0.002	-0.059	1.048	0.062	16.925	1.110	0.041	27.264
Wholesale	0.000	0.001	-0.255	1.089	0.050	21.710	1.119	0.035	32.173
Retail	0.001	0.002	-0.967	0.958	0.060	16.094	1.088	0.038	28.648
Restaurants & Hotels	0.002	0.002	-1.051	1.083	0.091	11.925	1.208	0.059	20.582
Banking, Ins., Real Estate	0.001	0.001	-0.528	1.002	0.044	22.579	1.010	0.030	34.162
Other	-0.001	0.001	0.896	1.075	0.054	19.783	1.089	0.037	29.397
								J Statistic	42.08447
								P Value	0.070391

The GMM estimates for the industry ET Betas are plotted against mean excess monthly return in Figure 6.

FIGURE 6

GMM Estimates - Industry Portfolio Betas

Figure 6 presents the ET Beta GMM estimates against the mean monthly excess return for each of the industry portfolios. The square indicates the CRSP market index. The industry portfolios are numbered 1 through 30. The definitions can be found in Appendix E.



The p-values obtained for other values of n are contained in Table 3 below. The p-value appears to peak somewhere close to $n = 2$, corresponding to third order partial stochastic dominance. This suggests that investors possess concave utility functions whose derivatives alternate in sign up to the third derivative.

The degree $n = 3$ corresponds to fourth order stochastic dominance, a subset of third order stochastic dominance. The degree $n = 4$ corresponds to fifth order stochastic dominance, a subset of fourth order stochastic dominance. These degree values can be thought of as measures of partial skewness and kurtosis.

TABLE 3
P-values under Varying Degrees for the ET-CAPM

Table 3 displays the J-statistics and p-values for the ET-CAPM for several degrees. The p-value appears to peak somewehere close to degree 2. A Bonferroni correction lowers the 25% rejection threshold to 0.042; the model is not rejected at this level.

Degree	1.5	2.0	2.5	3.0	3.5	4.0
J-Statistic	42.25	42.08	42.15	42.55	43.24	44.06
P-Value	0.068	0.070	0.069	0.064	0.056	0.047

Table 3 constitutes six tests of the ET-CAPM. Under a Bonferroni correction for multiple tests the 25% rejection level corresponds to a p-value of 0.042. Thus the model fails to reject at this level.

The EL-CAPM can be thought of as the partial moment analogue of the SL-CAPM. However the ET-CAPM is the partial moment analogue[16] of the Black CAPM; the zero-β asset is not correlated with the market portfolio. Here I test the Black CAPM directly on the same data set using the following moment restrictions

$$g_T(\phi) = \frac{1}{T}\sum_{t=1}^{T}\begin{bmatrix} \varepsilon_j - R_{j,t}^z + {}^{BL}\beta_j\, R_{m,t}^z \\ {}^{BL}\beta_j\left({R_{m,t}}^2 - {\mu_m}^2\right) - R_{m,t}R_{j,t} - \mu_m R_{j,t} \\ R_{m,t} - \mu_m \end{bmatrix} \quad for\ j = 1\ to\ 30 \qquad [51]$$

where $\phi = (\varepsilon_1, \cdots, \varepsilon_{30}, {}^{BL}\beta_1, \cdots, {}^{BL}\beta_{30}, \mu_m^z)$ are the parameters to be estimated. The second and third lines of these restrictions do not involve excess returns as the beta in the Black CAPM

[16] Harlow and Rao (1989) apply an asymmetric response model in order to 'allow for the testing of a more general Black (1972) –type MLPM (EL) model in which the risk free rate is replaced with a zero beta asset'. Their zero-beta MLPM model is not a an EL analogue of the Black CAPM. The following should be noted: (a) Harlow and Rao use an unconventional formulation for both the LPM and CLPM in their model without any economic justification: $LPM_{HR}(R_p, R_f, h) = \int_{-\infty}^{h}(h - R_p)(R_f - R_p)f(R_p)dR_p$. Their formulation involves both an unspecified threshold rate and the risk-free rate (later replaced with a zero-beta rate), (b) their zero-beta approach is based on the mutual fund separation arguments of Ross (1978); they assume multivariate normality in order to achieve separation – a very significant loss of generality, (c) there is no analytical derivation of their zero-beta model. With the necessary assumption of multivariate normality and the unusual LPM and CLPM formulations their model falls somewhere between the SL-CAPM and Black CAPM.

involves actual returns. Typically the Black CAPM is tested without specifying the zero-β asset; for the purposes of comparison the monthly return generated by investing at the daily effective Federal Funds rate is again used. The null hypothesis is

$$H_0: \bar{\varepsilon} = \mathbf{0} \qquad [52]$$

The moment restrictions constitute 61 equations with 31 unknowns; the 30 industry betas and the market mean. The Newey West difference test for over-identifying restrictions is conducted by setting $\bar{\varepsilon} = \mathbf{0}$ and calculating the difference statistic. The results contained within Table 4 indicate that the Black CAPM is rejected at the 5% significance level (p-value of 0.04)[17].

[17] A similar test of the ET-CAPM and Black CAPM over the same period was conducted using the monthly returns from the Fama-French 61-120 month bond portfolio as the zero-β asset. The results were p-values of 0.022 and 0.009 respectively.

TABLE 4

Black CAPM - GMM Estimation of Parameters in Unrestricted Model and Test of Overindentifying Restrictions

Table 4 displays the parameter estimates (epsilons, betas and market mean) from the unrestricted and restricted versions of the Black CAPM. The J statistic and p-value (0.041965) for the Null implies that the model is rejected at the 5% level. The data spans July-1954 to December-2008.

	Unrestricted Model						Restricted Model		
	Epsilon			Betas and Mean			Betas and Mean		
Industry Portfolio	**Estimate**	**Standard Error**	**T-Ratio**	**Estimate**	**Standard Error**	**T-Ratio**	**Estimate**	**Standard Error**	**T-Ratio**
Food	0.003	0.001	2.245	0.740	0.043	17.398	0.800	0.037	21.821
Beer & Liquor	0.003	0.002	1.926	0.797	0.053	15.095	0.855	0.046	18.467
Tobacco Products	0.006	0.002	2.654	0.683	0.060	11.419	0.789	0.052	15.182
Recreation	0.000	0.002	0.162	1.260	0.051	24.823	1.267	0.045	28.097
Publishing	0.000	0.001	0.087	1.029	0.047	21.917	1.052	0.039	27.109
Consumer Goods	0.002	0.001	1.378	0.841	0.042	20.091	0.923	0.036	25.490
Apparel	0.000	0.002	0.029	1.059	0.055	19.140	1.102	0.049	22.343
Healthcare	0.003	0.001	2.412	0.881	0.046	19.060	0.933	0.041	22.889
Chemicals	-0.001	0.001	-0.452	1.004	0.037	26.968	1.061	0.030	35.299
Textiles	0.000	0.002	-0.025	1.003	0.057	17.446	1.045	0.047	22.017
Construction	0.000	0.001	-0.232	1.132	0.035	32.509	1.165	0.028	41.450
Steel	-0.002	0.002	-1.017	1.248	0.058	21.570	1.207	0.050	24.023
Fabricated Products	0.000	0.001	-0.292	1.163	0.035	33.474	1.157	0.029	39.738
Electrical	0.001	0.001	1.178	1.179	0.032	36.286	1.219	0.029	41.638
Automobiles	-0.001	0.002	-0.716	1.048	0.057	18.538	1.048	0.045	23.131
Transport Equipment	0.001	0.002	0.785	1.083	0.048	22.459	1.132	0.041	27.373
Mining	0.001	0.002	0.211	0.911	0.071	12.907	0.893	0.057	15.680
Coal	0.005	0.004	1.264	1.134	0.093	12.183	1.115	0.074	15.091
Oil	0.003	0.002	1.828	0.800	0.044	18.140	0.811	0.040	20.078
Utilities	0.002	0.001	1.300	0.558	0.040	13.854	0.615	0.035	17.638
Communication	0.001	0.001	0.470	0.743	0.038	19.496	0.714	0.033	21.519
Services	0.001	0.001	0.603	1.272	0.041	30.976	1.251	0.036	34.724
Business Equipment	0.000	0.002	0.273	1.286	0.049	26.312	1.245	0.042	29.822
Paper	0.000	0.001	0.032	0.945	0.032	29.563	1.003	0.027	37.478
Transportation	0.000	0.001	0.055	1.052	0.040	26.567	1.094	0.036	30.736
Wholesale	0.000	0.001	0.344	1.068	0.037	28.549	1.084	0.033	32.656
Retail	0.001	0.001	0.985	0.987	0.041	23.950	1.050	0.034	30.477
Restaurants & Hotels	0.002	0.002	1.184	1.072	0.058	18.366	1.135	0.052	21.821
Banking, Ins., Real Estate	0.001	0.001	0.547	1.010	0.029	34.341	1.007	0.026	39.234
Other	-0.001	0.001	-0.882	1.046	0.037	28.303	1.078	0.033	32.984
				0.004	0.002	2.213	0.005	0.002	2.913
								J Statistic	44.60809
								P Value	0.041965

Figure 7 below shows the close relationship between ET-CAPM betas and Black CAPM betas. Figure 8 presents the Beta and alpha estimates for the ET-CAPM. It also presents Beta and epsilon estimated for the Black CAPM. The least error occurs for the industry portfolios that are closely correlated with the market index. A positive error is associated with lower correlation and vice versa.

FIGURE 7

Comparison between the ET-CAPM and Black CAPM Beta Estimates

Figure 7 presents the ET-CAPM and Black CAPM Beta estimates for the industry portfolios. The portfolios that have the largest absolute beta difference (>0.05) are Recreation, Restaurants & Hotels, Textiles and Mining. The portfolios with the least absolute beta difference (<0.005) are Healthcare, Services, Utilities, Paper, Banking, Insurance & Real Estate and Construction.

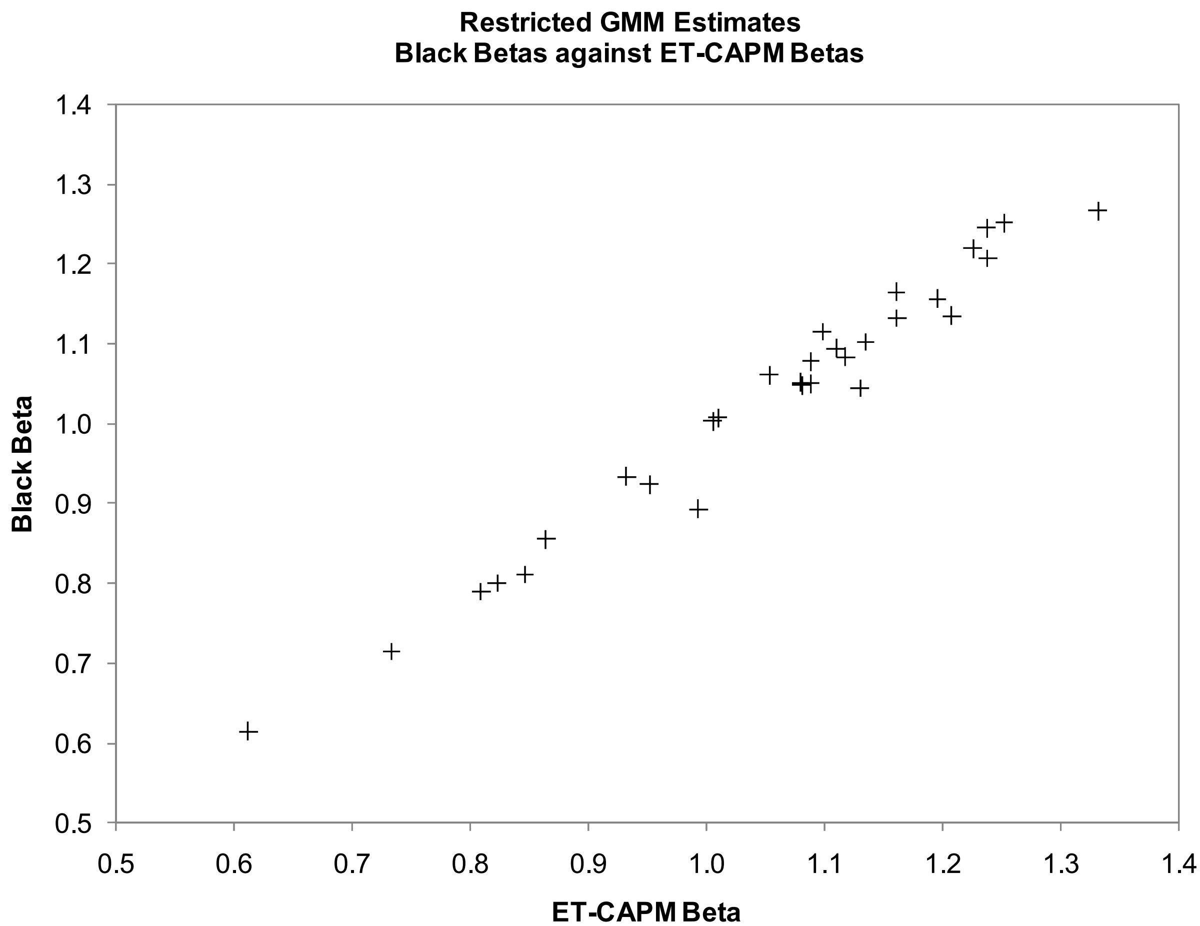
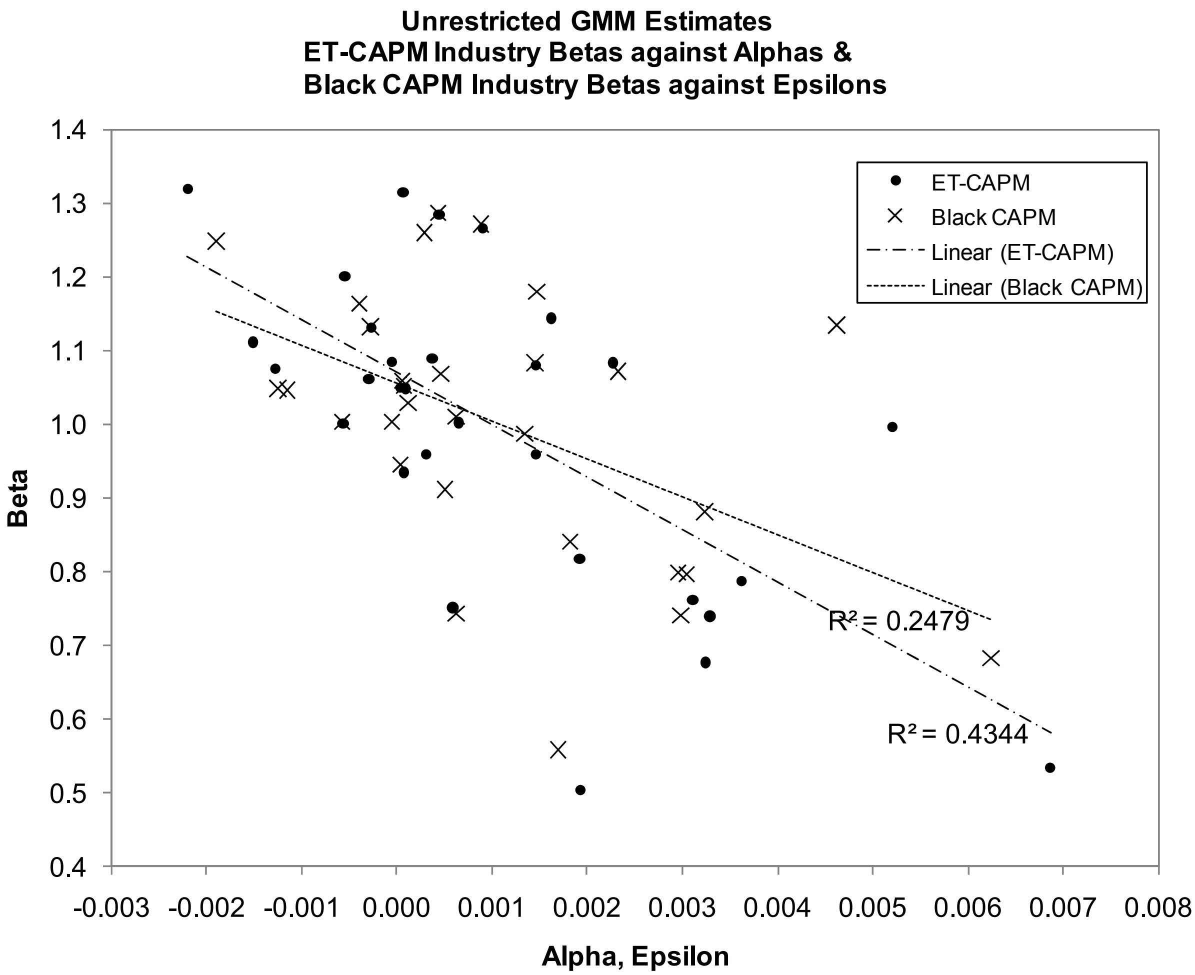


FIGURE 8

Comparison between the ET-CAPM and Black CAPM Beta Estimates

Figure 8 presents the Beta and alpha estimates for the ET-CAPM. It also presents Beta and epsilon estimated for the Black CAPM. The least error occurs for the industry portfolios that are closely correlated with the market index. A positive error is associated with lower correlation and vice versa.



X. Conclusion

The EL-CAPM of Bawa and Lindenberg (1977) has the attractive property of being motivated by stochastic dominance and utility theory. The framework upon which it rests however is questionable from an asset pricing perspective; the zero-LPM asset is the relevant risk-free asset in the EL framework however portfolio separation does not obtain between this asset and a risky portfolio. Although well motivated, the model rests upon singular stochastic dominance.

In this paper the ET asset pricing framework was introduced and several portfolio separation results were derived. In particular, a framework consistent zero-β separation result was derived. Previous mean and monetary EL separation results were shown to be special cases of ET separation results. From the zero-β separation result the ET-CAPM was derived. Like the EL-CAPM, this model is motivated by stochastic dominance and utility theory however it rests on partial stochastic dominance.

This model also has two desirable characteristics; the return distributions for all assets are arbitrary (beyond LPM existence) and there is a direct link between the degree of the model and the class of utility function. It was shown that the ET-CAPM could not be readily extended to accommodate divergent borrowing and lending facilities.

As a demonstration the model was applied to the 30 Fama-French industry portfolios and the return generated from investing in the daily effective federal funds rate. This application underlies a key difference between the ET and EL capital asset pricing models. The ET-CAPM is capable of representing the situation faced by many fund managers and investors; investment allocations being made on a periodic basis whilst maintaining some portion in overnight cash.

An empirical test of the model with $n = 2$ was conducted using the generalized method of moments. Unlike the Black CAPM, the ET-CAPM could not be rejected at the 5% significance level. This corresponds to third order partial stochastic dominance and suggests that investors possess concave utility functions whose derivatives alternate in sign up to the third derivative. Furthermore, under a Bonferroni adjustment the ET-CAPM could not be rejected at the 25% level of significance.

Appendix A

Proof of Theorem 5 (Mean Separation):

Consider a mix of portfolios such that

$$R_q = \mathbf{x}^{\mathrm{T}}\mathbf{1}R_p + (1-\ \mathbf{x}^{\mathrm{T}}\mathbf{1})R_z \qquad [A1]$$

where R_z is the stochastic return on the zero-β asset. The normalized TPWL with threshold $h = \mathbf{x}^{\mathrm{T}}\mathbf{1}E\left[R_p\right] + \left(1-\ \mathbf{x}^{\mathrm{T}}\mathbf{1}\right)R_z$ is

$$TPWL_n^{1/n}\left(R_q, h = \mathbf{x}^{\mathrm{T}}\mathbf{1}E\left[R_p\right] + (1-\ \mathbf{x}^{\mathrm{T}}\mathbf{1})R_z\right)$$

$$= \left(\int_a^b \left[\int_{-\infty}^{\mathbf{x}^{\mathrm{T}}\mathbf{1}E[R_p]+(1-\mathbf{x}^{\mathrm{T}}\mathbf{1})R_z} \left(\mathbf{x}^{\mathrm{T}}\mathbf{1}E\left[R_p\right] + (1-\ \mathbf{x}^{\mathrm{T}}\mathbf{1})R_z - R_q\right)^n c\left(R_q\right)dR_q\right] dR_z\right)^{\frac{1}{n}} \qquad [A2]$$

Apply the following transformation of variables.

$$R_q = \mathbf{x}^{\mathrm{T}}\mathbf{1}R_p + (1-\ \mathbf{x}^{\mathrm{T}}\mathbf{1})R_z \qquad \text{so} \qquad dR_q = \mathbf{x}^{\mathrm{T}}\mathbf{1}dR_p$$

$$R_p = \frac{1}{\mathbf{x}^{\mathrm{T}}\mathbf{1}}\left[R_q - (1-\ \mathbf{x}^{\mathrm{T}}\mathbf{1})R_z\right]$$

$$\text{and} \qquad c\left(R_q\right) = \mathcal{J}c\left(R_p, R_z\right) \text{ with } \mathcal{J} = \frac{dR_p}{dR_q} = \frac{1}{\mathbf{x}^{\mathrm{T}}\mathbf{1}}$$

By definition R_z is independent of R_p and thus their joint distribution can be written as $c(R_p, R_z) = f(R_p)g(R_z)$. Under this transformation the upper limit of the integral becomes $\frac{1}{\mathbf{x}^{\mathrm{T}}\mathbf{1}}\left(E\left[R_q\right] - \left(1-\ \mathbf{x}^{\mathrm{T}}\mathbf{1}\right)R_z\right) = E\left[R_p\right]$.

$$TPWL_n^{1/n}\left(R_q, h = \mathbf{x}^{\mathrm{T}}\mathbf{1}E\left[R_p\right] + (1-\ \mathbf{x}^{\mathrm{T}}\mathbf{1})R_z, a, b\right)$$

$$= \left(\int_a^b \left[\int_{-\infty}^{E[R_p]} \begin{pmatrix} \mathbf{x}^{\mathrm{T}}\mathbf{1}E\left[R_p\right] + (1-\ \mathbf{x}^{\mathrm{T}}\mathbf{1})R_z \\ -\mathbf{x}^{\mathrm{T}}\mathbf{1}R_p - (1-\ \mathbf{x}^{\mathrm{T}}\mathbf{1})R_z \end{pmatrix}^n f\left(R_p\right)g(R_z)dR_p\right] dR_z\right)^{\frac{1}{n}} \qquad [A3]$$

$$= \mathbf{x}^{\mathrm{T}}\mathbf{1}\left(\int_a^b \left[\int_{-\infty}^{E[R_p]} \left(E[R_p] - R_p\right)^n f(R_p) g(R_z) dR_p\right] dR_z\right)^{\frac{1}{n}} \qquad [A4]$$

$$= \mathbf{x}^{\mathrm{T}}\mathbf{1}TPWL_n^{1/n}\left(R_p, h = E[R_p]; a, b\right) \qquad [A5]$$

■

Appendix B

The gradient of the curve pj at q is given by

$$\frac{dE[R_q]}{dTPWL_n^{1/n}[R_q,R_z]} = \left[\frac{dE[R_q]}{d\mathbf{x}^{\mathrm{T}}\mathbf{1}}\right]\left[\frac{dTPWL_n^{1/n}[R_q,R_z]}{d\mathbf{x}^{\mathrm{T}}\mathbf{1}}\right]^{-1} \qquad [B1]$$

The first term on the RHS is readily calculated:

$$\left[\frac{dE[R_q]}{d\mathbf{x}^{\mathrm{T}}\mathbf{1}}\right] = E[R_j] - E[R_p] \qquad [B2]$$

The second term is a little more difficult:

$$\frac{dTPWL_n^{1/n}[R_q,R_z]}{d\mathbf{x}^{\mathrm{T}}\mathbf{1}} = \frac{1}{n}\left(E\left[(R_z-R_q)^{+n}\right]\right)^{\frac{1-n}{n}} nE\left[-(R_z-R_q)^{+(n-1)}(R_j-R_p)\right] \qquad [B3]$$

$$= \left(E\left[(R_z-R_q)^{+n}\right]\right)^{\frac{1-n}{n}} E\left[(R_z-R_q)^{+(n-1)}(R_p-R_j)\right] \qquad [B4]$$

So

$$\frac{dE[R_q]}{dTPWL_n^{1/n}[R_q,R_z]} = \left[\left(E\left[(R_z-R_q)^{+n}\right]\right)^{\frac{1-n}{n}} E\left[(R_z-R_q)^{+(n-1)}(R_p-R_j)\right]\right]^{-1} E[R_j-R_p] \qquad [B5]$$

At the tangent point p the portfolio q is entirely in portfolio p and $\mathbf{x}^{\mathrm{T}}\mathbf{1} = 0$. So I can write:

$$\left.\frac{dE[R_q]}{dTPWL_n^{1/n}[R_q,R_z]}\right|_{\mathbf{x}^{\mathrm{T}}\mathbf{1}=0} = \left[\left(E\left[(R_z-R_p)^{+n}\right]\right)^{\frac{1-n}{n}} E\left[(R_z-R_p)^{+(n-1)}(R_p-R_j)\right]\right]^{-1} E[R_j-R_p] \qquad [B6]$$

$$= \left(TPWL_n^{1/n}[R_p,R_z]TPWL_n^{-1}[R_p,R_z]E\left[(R_z-R_p)^{+(n-1)}(R_p-R_j)\right]\right)^{-1} E[R_j-R_p] \qquad [B7]$$

But from the tangent line the gradient is:

$$\frac{dE[R_p]}{dTPWL_n^{1/n}[R_p, R_z]} = \frac{E[R_p - R_z]}{TPWL_n^{1/n}[R_p, R_z]} \qquad [B8]$$

Equating these gradients gives:

$$E[R_j - R_p] = E[R_p - R_z]\frac{E\left[(R_z - R_p)^{+(n-1)}(R_p - R_j)\right]}{TPWL_n[R_p, R_z]} \qquad [B9]$$

Now replace $(R_p - R_j)$ with $(R_p - R_z) + (R_z - R_j)$

$$E[R_j - R_p] = \frac{E[R_p - R_z]}{TPWL_n[R_p, R_z]}\, E\left[(R_z - R_p)^{+(n-1)}[(R_p - R_z) + (R_z - R_j)]\right] \qquad [B10]$$

$$= \frac{E[R_p - R_z]}{TPWL_n[R_p, R_z]}\left(-TPWL_n[R_p, R_z] + E\left[(R_z - R_p)^{+(n-1)}(R_z - R_j)\right]\right)$$

$$= -E[R_p - R_z] + E[R_p - R_z]\frac{E\left[(R_z - R_p)^{+(n-1)}(R_z - R_j)\right]}{TPWL_n[R_p, R_z]}$$

$$E[R_j - R_p] + E[R_p - R_z] = E[R_p - R_z]\frac{E\left[(R_z - R_p)^{+(n-1)}(R_z - R_j)\right]}{TPWL_n[R_p, R_z]} \qquad [B11]$$

$$E[R_j - R_z] = {}^{ET}_{n}\beta_j E[R_p - R_z] \qquad n \geq 1 \qquad [B12a]$$

$${}^{ET}_{n}\beta_j = \frac{CTPWL_n[R_j, R_p, R_z]}{TPWL_n[R_p, R_z]} \qquad [B12b]$$

$$CTPWL_n[R_j, R_p, R_z] = E\left[(R_z - R_p)^{+(n-1)}(R_z - R_j)\right] \qquad [B12c]$$

$$TPWL_n[R_p, R_z] = E\left[(R_z - R_p)^{+n}\right] \qquad [B12d]$$

Appendix C

In this appendix the lemma and theorem of section VI are proven

Lemma: If the following expression holds between three random variables X, Y and Z

$$\frac{Cov\left(X^{+^{n-1}},Y\right)}{Cov\left(X^{+^{n-1}},Z\right)}=\frac{E(Y)}{E(Z)} \qquad [C1]$$

then

$$\frac{CLPM_n(X,Y)}{CLPM_n(X,Z)}=\frac{Cov\left(X^{+^{n-1}},Y\right)}{Cov\left(X^{+^{n-1}},Z\right)}=\frac{E(Y)}{E(Z)} \qquad [C2]$$

Proof: The LHS of $[C1]$ can be rewritten to form

$$\frac{E\left(X^{+^{n-1}}Y\right)-E\left(X^{+^{n-1}}\right)E(Y)}{E\left(X^{+^{n-1}}Z\right)-E\left(X^{+^{n-1}}\right)E(Z)}=\frac{E(Y)}{E(Z)} \qquad [C3]$$

Rearranging yields:

$$E\left(X^{+^{n-1}}Y\right)E(Z)-E\left(X^{+^{n-1}}\right)E(Y)E(Z)=E\left(X^{+^{n-1}},Z\right)E(Y)-E\left(X^{+^{n-1}}\right)E(Z)E(Y)$$

and cancellation gives

$$E\left(X^{+^{n-1}}Y\right)E(Z)=E\left(X^{+^{n-1}},Z\right)E(Y) \qquad [C4]$$

That is

$$\frac{CLPM_n(X,Y)}{CLPM_n(X,Z)}=\frac{E(Y)}{E(Z)} \qquad [C5]$$

■

Theorem: If the following expression holds between three random variables X, Y and Z and a fourth uncorrelated variable ε

$$\frac{Cov\left((X-\varepsilon)^{+^{n-1}},Y-\varepsilon\right)}{Cov\left((X-\varepsilon)^{+^{n-1}},Z-\varepsilon\right)}=\frac{E(Y-\varepsilon)}{E(Z-\varepsilon)} \qquad [C6]$$

then

$$\frac{CTPWL_n(X,Y,\varepsilon)}{CTPWL_n(X,Z,\varepsilon)}=\frac{Cov\left((X-\varepsilon)^{+^{n-1}},Y-\varepsilon\right)}{Cov\left((X-\varepsilon)^{+^{n-1}},Z-\varepsilon\right)}=\frac{E(Y-\varepsilon)}{E(Z-\varepsilon)} \qquad [C7]$$

Proof: The LHS of $[C6]$ can be rewritten to form

$$\frac{E\left((X-\varepsilon)^{+n-1}(Y-\varepsilon)\right)-E\left((X-\varepsilon)^{+n-1}\right)E(Y-\varepsilon)}{E\left((X-\varepsilon)^{+n-1}(Z-\varepsilon)\right)-E\left((X-\varepsilon)^{+n-1}\right)E(Z-\varepsilon)}=\frac{E(Y-\varepsilon)}{E(Z-\varepsilon)} \qquad [C8]$$

Rearranging gives

$$\begin{aligned} &E\left((X-\varepsilon)^{+n-1}(Y-\varepsilon)\right)E(Z-\varepsilon)-E\left((X-\varepsilon)^{+n-1}\right)E(Y-\varepsilon)E(Z-\varepsilon) \\ &\quad = E\left((X-\varepsilon)^{+n-1}(Z-\varepsilon)\right)E(Y-\varepsilon)-E\left((X-\varepsilon)^{+n-1}\right)E(Z-\varepsilon)E(Y-\varepsilon) \end{aligned}$$

and cancellation yields

$$E\left((X-\varepsilon)^{+n-1}(Y-\varepsilon)\right)E(Z-\varepsilon)=E\left((X-\varepsilon)^{+n-1}(Z-\varepsilon)\right)E(Y-\varepsilon) \qquad [C9]$$

That is

$$\frac{CTPWL_n(X,Y,\varepsilon)}{CTPWL_n(X,Z,\varepsilon)}=\frac{E(Y-\varepsilon)}{E(Z-\varepsilon)} \qquad [C10]$$

■

Appendix D

FIGURE D1

Efficient Frontier in ET Space $(n = 5)$

Figure D1 presents the efficient frontier in ET space. The marks on the return axis indicate the upper and lower pounds of the zero-β asset. The circle on the frontier indicates the tangent portfolio. The square indicates the market portfolio. The industry portfolios are numbered 1 through 30. The definitions can be found in Appendix E.

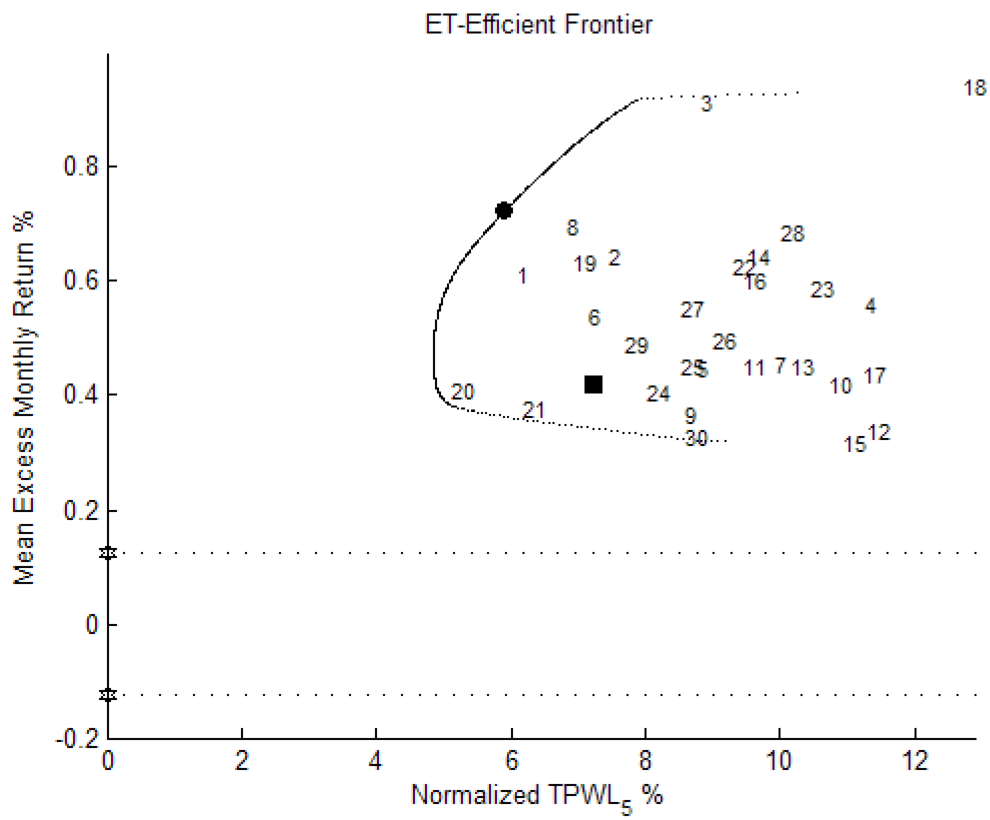


FIGURE D2

Industry Portfolio Tangent Betas $(n = 5)$

Figure D2 presents the tangent ET Betas against the mean monthly excess return for each of the industry portfolios. The disc indicates the location of the tangent portfolio. The industry portfolios are numbered 1 through 30. The definitions can be found in Appendix E.

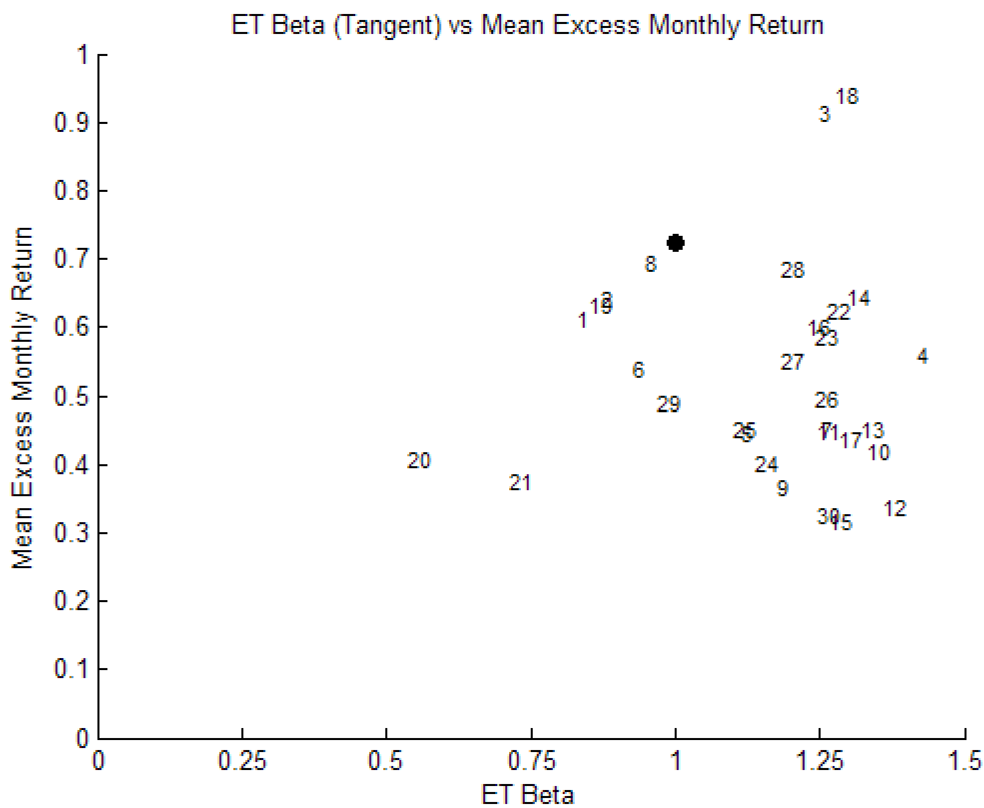
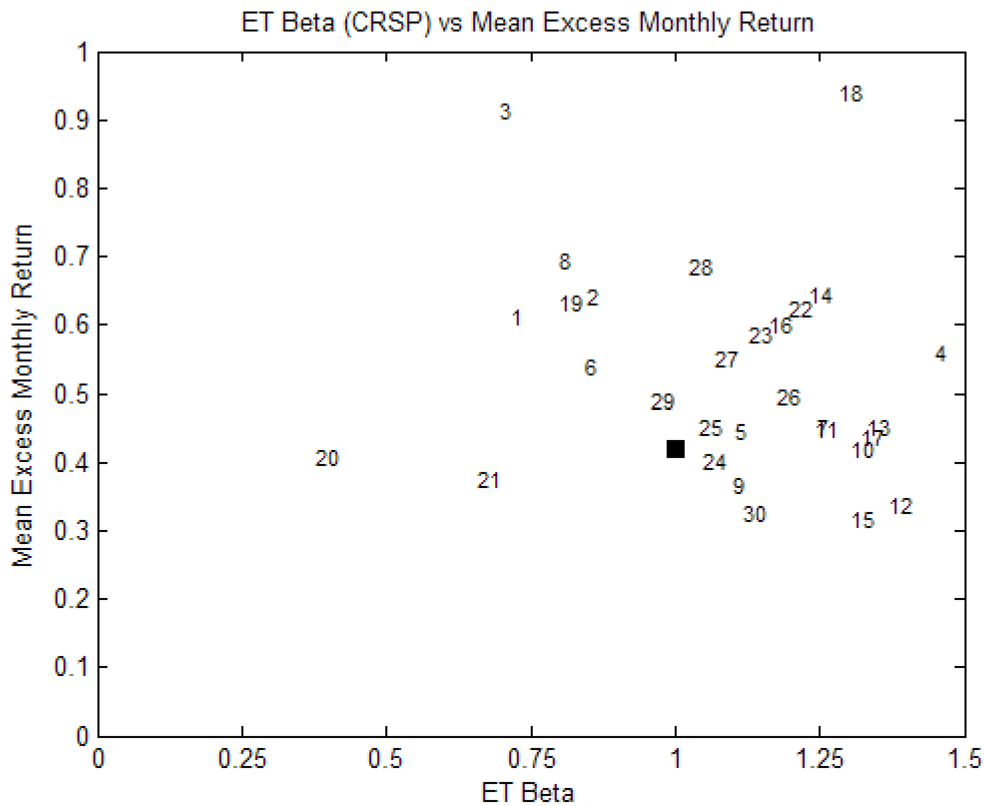


FIGURE D3

Industry Portfolio CRSP Index Betas $(n = 5)$

Figure D3 presents the CRSP Market ET Betas against the mean monthly excess return for each of the industry portfolios. The Square indicates the CRSP market index. The industry portfolios are numbered 1 through 30. The definitions can be found in Appendix E.



Appendix E

The table below provides the definitions for the Fama-French industry portfolios. More detailed information can be found at:

http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/Data_Library/det_30_ind_port.html

Industry Portfolio Number	**Name**
1	Food Products
2	Beer & Liquor
3	Tobacco Products
4	Recreation
5	Printing and Publishing
6	Consumer Goods
7	Apparel
8	Healthcare, Medical Equipment, Pharmaceutical Products
9	Chemicals
10	Textiles
11	Construction and Construction Materials
12	Steel Works
13	Fabricated Products and Machinery
14	Electrical Equipment
15	Automobiles and Trucks
16	Aircraft, ships, and railroad equipment
17	Precious Metals, Non-Metallic, and Industrial Metal Mining
18	Coal
19	Petroleum and Natural Gas
20	Utilities
21	Communication
22	Personal and Business Services
23	Business Equipment
24	Business Supplies and Shipping Containers
25	Transportation
26	Wholesale
27	Retail
28	Restaurants, Hotels, Motels
29	Banking, Insurance, Real Estate, Trading
30	Other

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