Defaultable HJM Term Structure Models with Stochastic Volatility.

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Abstract

In this paper, we present a class of defaultable term structure models within the HJM framework with stochastic volatility. Under certain volatility specifications, the model admits finite dimensional Markovian structures and consequently provides tractable solutions for interest rate derivatives. We also investigate the effect of stochastic volatility and correlation on the defaultable short rate and bond price.

Key Words: Stochastic volatility, Heath-Jarrow-Morton, defaultable forward rates, credit spreads.

1 Introduction

The Heath, Jarrow and Morton (1992) framework (hereafter referred to as HJM) is considered the most general and flexible setting for the study of interest rates dynamics and pricing of interest rate derivatives. The model is automatically calibrated to the currently observed yield curve and is complete as it does not involve the market price of risk, a feature common in earlier generations of interest rate models such as Vasicek (1977) and Cox, Ingersoll and Ross (1985).

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The input required for the model is the currently observed forward rate curve and the volatility structure of the forward interest rates. The no riskless arbitrage condition implies that the drift coefficient of the forward rate dynamics is expressed in terms of the forward rate volatility function. However, since the initial forward rate is completely determined by the market, the only remaining flexibility for obtaining finite dimensional Markovian models within the HJM framework rests in a pertinent specification of the volatility function. Various restrictions on the forward rate volatility that lead to a finite dimensional Markovian HJM models were obtained in Ritchken and Sankarasubramanian (1995), Bhar and Chiarella (1997) and Inui and Kijima (1998) under diffusion processes. Extensions to jump-diffusions has been studied by Chiarella and Nikitopoulos (2003).

The modelling of the defaultable term structure using the HJM model was first examined by Jarrow and Turnbull (1995) and Duffie and Singleton (1999). Schönbucher (1998) showed that a model of the spread of the defaultable interest rates over the default free interest rates may be used to add a default-risk module to an existing model of default-free interest rates. Various forms of the no arbitrage condition between the default free and the defaultable term structures were derived from which the term structure of defaultable bond prices was then obtained. The model developed assumed that a jump in the defaultable forward rate leads to default. In addition, he showed that the forward rate credit spread offers the link between the defaultable and default free term structures.

Maksymiuk and Gatarek (1999) obtained the HJM condition for the forward credit spread. They showed that under zero recovery rate and assuming no correlation between default and risk-free rates, the initial spread term structure coincides with the initial term structure of the intensity process. These results were extended by Pugachevsky (1999) to allow cases of non-zero correlation. He showed that the risky forward rate is the sum of risk-free forward rate and spread and also derived the relationship between the drift and volatility terms for the spread between forward rates.

Chiarella, Nikitopoulos and Schlögl (2007) modelled the defaultable forward rate as a jump-diffusion process where the jumps in the defaultable term structure cause jumps and defaults to the defaultable bond prices. A forward rate volatility structure was investigated that results in a Markovian defaultable short rate. Empirical evidence given in D'Souza, Amir-Atefi and Racheva-Jotova (2004) show that short rate and intensity processes exhibit stochastic volatility which generates the heavy-tailed behaviour observed in their unconditional distribution of daily movements.

There are two main contributions we make in this paper: First, we incorporate stochastic volatility within the generalised framework developed in Schönbucher (1998), although we do not require the defaultable forward rate curve to have a jump component. Secondly, we investigate the problem of Markovianisation within the developed defaultable term structure model with stochastic volatility. Most of the existing literature focusses on Markovianisation in the default free set up and while Chiarella et al. (2007) addressed the problem in the defaultable setup, the assumption of constant volatility was used.

The structure of this paper is as follows: In Section 2 we review the defaultable term structure developed in Schönbucher (1998) which we then generalise to allow for a correlation structure between forward rate, forward credit spread and the stochastic volatility. In Section 3, we assume specific volatility structures and derive Markovian representation of the defaultable short rate in terms of a finite number of state variables. We then derive an explicit bond price formula as a function of the state variables and present some simulation results on the distributional properties of the defaultable bond price and bond returns. In Section 4, we express the state variables as finite dimensional realisations in terms of economic quantities observed in the market, in particular in terms of the forward rates. Section 5 discusses the generalised defaultable HJM structure and Section 6 concludes the paper.

2 Multi-Factor Heath-Jarrow-Morton Framework

The model is setup on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ where $\mathbb{P}$ is the real world probability measure and filtration $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^N$, $t \geq 0$ satisfies the usual conditions. The subfiltration $\mathcal{F}_t^W$ is generated by the standard $\mathbb{P}-$Wiener process $W(t)$,

$$(\mathcal{F}_t^W)_{t\geq 0} = \{\sigma(W_s : 0 \leq s \leq t)\}_{t\geq 0}, \tag{2.1}$$

and represents the flow of all background information except from default itself which generates the sub-filtration $\mathcal{F}_t^N$,

$$(\mathcal{F}_t^N)_{t\geq 0} = \{\sigma(\mathbb{1}_{\{\tau\leq s\}} : 0 \leq s \leq t)\}_{t\geq 0}. \tag{2.2}$$

The instantaneous forward interest rate prevailing at time t for the maturity $T \geq t$ is denoted by $f(t,T,\omega)$ and is defined as

$$f(t,T,\omega) = -\frac{\partial}{\partial T}P(t,T), \quad \text{for all} \quad t \in [0,T]. \tag{2.3}$$

This is the corresponds to the rate that one can contract for at time t for instantaneous borrowing at time T. Solving the differential equation in (2.3) yields

$$P(t,T) = \exp\Big(-\int_t^T f(t,s,\omega)ds\Big), \quad 0 \leq t \leq s \leq T. \tag{2.4}$$

The multi-factor Heath et al. (1992) term structure model specifies the dynamics of the forward rate curve by the stochastic integral equation (hereafter SIE)

$$f(t,T,\omega) = f(0,T) + \int_0^t \alpha^f(u,T,\omega)du + \sum_{i=1}^n \int_0^t \sigma_i^f(u,T,\omega)dW_i^f(u), \quad 0 \leq u \leq T, \tag{2.5}$$

where the n noise terms dW_i are the increments of independent Wiener processes, $\alpha^f(u,T,\omega)$ and $\sigma_i^f(u,T,\omega)$ are the drift and volatility terms of the forward rate process and $\omega \in \Omega$ represents the dependence of the forward rate process on the Wiener path. Let $P(t,T)$ denote the time t price of the $T-$maturity default-free discount bond with $0 \leq t \leq T$. The default-free short rate at time t, $r(t,\omega)$ is the instantaneous forward rate at time t for date t, $r(t,\omega) = f(t,t,\omega)$. Setting $T = t$ in equation (3.8a), the short rate process is shown to follow the stochastic integral equation

$$r(t,\omega) = f(0,t) + \int_0^t \alpha^f(u,t,\omega)du + \sum_{i=1}^n \int_0^t \sigma_i^f(u,t,\omega)dW_i^f(u), \quad 0 \leq u \leq t. \tag{2.6}$$

Heath et al. (1992) proved that the absence of risk-less arbitrage opportunities implies that the drift term cannot be chosen arbitrarily but rather will be some function of the volatility function and the market prices of interest rate risk. This is the HJM no-arbitrage drift restriction condition

$$\alpha^f(t,T,\omega) = -\sum_{i=1}^n \sigma_i^f(t,T,\omega)\Big(\phi_i(t) - \int_t^T \sigma_i^f(t,s,\omega)ds\Big), \tag{2.7}$$

where $\phi_i(t)$ denotes the market price of interest rate risk associated with the noise process $W_i^f(t)$.

Furthermore, by an application of Girsanov's theorem there exists a risk-neutral measure $\tilde{\mathbb{P}}$ such that

$$d\tilde{W}_i^f(t) = dW_i^f(t) - \phi_i(t)dt.$$

The explicit dependence on the market prices of risk can be suppressed and the arbitrage-free stochastic integral equation for the default-free forward rate can be written as

$$f(t,T,\omega) = f(0,T) + \sum_{i=1}^{n} \int_0^t \sigma_i^f(u,T,\omega) \int_u^T \sigma_i^f(u,s,\omega) ds du + \sum_{i=1}^{n} \int_0^t \sigma_i^f(u,T,\omega) d\tilde{W}_i^f(u). \tag{2.8}$$

In addition, the default-free short rate dynamics in (2.6) satisfy under the risk-neutral measure $\tilde{\mathbb{P}}$ the SIE

$$r(t,\omega) = f(0,t) + \sum_{i=1}^{n} \int_0^t \sigma_i^f(u,t,\omega) \int_u^t \sigma_i^f(u,s,\omega) ds du + \sum_{i=1}^{n} \int_0^t \sigma_i^f(u,t,\omega) d\tilde{W}_i^f(u). \tag{2.9}$$

It is therefore observed that the dynamics of the term structure under the risk-neutral measure are entirely captured by the volatility functions. If we define some path dependent state variables $S_1(t,\omega)$ and $\psi_1(t,\omega)$ as

$$S_1(t,\omega) = \sum_{i=1}^{n} \int_0^t \sigma_i^f(u,t,\omega) \int_u^t \sigma_i^f(u,s,\omega) ds du, \tag{2.10}$$

$$\psi_1(t,\omega) = \sum_{i=1}^{n} \int_0^t \sigma_i^f(u,t,\omega) d\tilde{W}(u), \tag{2.11}$$

then equation (2.9) can be written as

$$r(t,\omega) = f(0,t) + S_1(t,\omega) + \psi_1(t,\omega), \tag{2.12}$$

which in differential form can be written as

$$dr(t,\omega) = \big[f_2(0,t) + \frac{\partial}{\partial t} S_1(t,\omega)\big] dt + d\psi_1(t,\omega). \tag{2.13}$$

This definition of the state variables will be useful later in equation (4.12) and Proposition 4.1.

Using equation (2.4), (2.8) and some stochastic manipulations including the use of Fubini's theorem and Itô's lemma, the bond price $P(t,T)$ must satisfy the stochastic differential equation

$$dP(t,T) = r(t,\omega)P(t,T)dt + \sum_{i=1}^{n} \tilde{\sigma}_{B,i}^f(t,T,\omega)P(t,T) d\tilde{W}_i^f(t), \tag{2.14}$$

where $\tilde{\sigma}_{B,i}^f(t,T,\omega) = -\int_t^T \sigma_i^f(t,s,\omega)ds$. The relative default-free bond price $Z(t,T,\omega) = P(t,T)(B(t,\omega))^{-1}$, where

$$B(t,\omega) = \exp\Big(\int_0^t r(s,\omega)ds\Big), \tag{2.15}$$

is the accumulated money market account and satisfies the stochastic differential equation

$$dZ(t,T) = Z(t,T)\sum_{i=1}^{n} \tilde{\sigma}^{f}_{B,i}(t,T,\omega)d\tilde{W}^{f}_{i}(t). \tag{2.16}$$

If we let $\tilde{\mathbb{E}}$ denote mathematical expectation with respect to the risk neutral probability measure, it then follows from equation (2.16) that

$$P(t,T) = \tilde{\mathbb{E}}\Big[\exp\Big(-\int_t^T r(s,\omega)ds\Big)\Big|\mathcal{F}_t\Big]. \tag{2.17}$$

More generally, if $C(t,T_C)$ is the price process for a T_C-expiry option on $P(t,T)$, with $T_C \le T$ and payoff $g(T_C)$, then

$$C(t,T_C) = \tilde{\mathbb{E}}\Big[e^{-\int_t^{T_C} r(s,\omega)ds} g(T_C)\Big|\mathcal{F}_t\Big]. \tag{2.18}$$

3 Defaultable HJM Term Structure

In this section, we introduce the defaultable HJM structure as a generalisation of Section 2 following the approach in Schönbucher (1998). We begin by introducing the default process and then show how multiple defaults and recoveries can be incorporated within the HJM framework when there is no jump within the forward rate dynamics. We then derive the no-arbitrage, drift restriction conditions under this defaultable framework. Finally, we deduce the fundamental bond pricing equation under the expectation operator within the risk-neutral measure.

3.1 Model Setup

Definition 3.1 *The process $N = \{N(t), t \ge 0\}$ is called a Cox process if there is a non-negative $\mathcal{F}_t-$adapted stochastic process $h(t)$ with*

$$\int_0^t h(s)ds < \infty \quad \forall t,$$

and conditional on the realization $\{h(t)\}_{t>0}$ of the intensity, $N(t)$ is a time inhomogeneous Poisson process with intensity $h(t)$.

The default time corresponds to the first jump of a Poisson process N which is characterized by a general intensity process $h(t)$. We assume that the default counting process

$$N(t) = \sum_{i=1}^{\infty} \mathbb{1}_{\{\tau_i \leq t\}},$$

is a counting process with intensity $h(t)$, where τ_i is the time of the *i-th* default. The increasing sequence of defaults are modelled by the counting process.

The intensity of the process is independent of previous defaults and therefore the time of default becomes a totally inaccessible stopping time. Since $N(t)$ is nondecreasing, $h(t)$ is nondecreasing, predictable and of finite variation so that we may define

$$\Lambda(t) = \int_0^t h(s)ds, \tag{3.1}$$

which is called the cumulative intensity process. The process $h(t)$ is the predictable compensator of $N(t)$ such that

$$M(t) := N(t) - \Lambda(t), \tag{3.2}$$

is a (purely discontinuous) martingale.

In practice, a default event does not terminate the debt contract as firms are usually reorganized and the debt is re-floated. This framework allows for subsequent defaults and hence multiple defaults are possible with the debt restructuring at each default event. The recovery rate $\mathcal{R}(t)$ given default is defined as the extent to which the value of an obligation can be recovered once the obligor has defaulted. This is a measure of the expected fractional recovery incase of default and therefore $\mathcal{R}(t) \in [0, 1]$.

In Duffie and Singleton (1997), the *fractional recovery of market value* model was developed where the compensation on default is made *in terms of equivalent defaultable bonds* which have not yet defaulted. In this case, recovery is expressed as a fraction of the defaulted bond just prior to default. This model was applied in Schönbucher (1998) and Schönbucher (2000). Duffie (1998) used *fractional recovery of par* where on default the compensation is made in cash invested in a non-defaultable money market account. Jarrow and Turnbull (1995) and Madan and Unal (1998) considered an alternative formulation, *fractional recovery of a default-free but otherwise equivalent bond*, (also called the equivalent recovery model) where on default, the compensation is made in terms of the value of non-defaultable bonds.

The increasing sequence of default times $\{\tau_i\}_{i\in\mathbb{N}}$ is driven by the Cox process. At each default time τ_i, the defaultable bond's face value is reduced by the loss rate $q(\tau_i)$, which can be a random variable. At maturity T, the defaultable bond subject to multiple defaults has a final payoff

$$\mathcal{R}(T) := \prod_{\tau_i \leq T} (1 - q(\tau_i)), \tag{3.3}$$

where $\mathcal{R}(T)$ is the product of the face reductions after all defaults until maturity T.

Definition 3.2 *1. Let $f^d(t,T,\omega)$ be the instantaneous defaultable forward rate of interest. Then, the price at time t of a defaultable zero coupon bond with maturity T is defined by*

$$P^d(t,T) = \mathcal{R}(t)\text{exp}\Big(- \int_t^T f^d(t,s,\omega)ds\Big). \tag{3.4}$$

2. In addition, we define the continuously compounded instantaneous forward credit spread as

$$\lambda(t,T,\omega) = f^d(t,T,\omega) - f(t,T,\omega). \tag{3.5}$$

The fraction $\mathcal{R}(t)$ represents the reduction on the bond's face value due to the number $N(t)$ of defaults up to time t. The pre-default price $\bar{P}^d(t,T,\omega)$ at time t of a defaultable zero-coupon with maturity T, a so-called 'pseudo' bond, is given by

$$\bar{P}^d(t,T,\omega) = \text{exp}\Big(- \int_t^T f^d(t,s,\omega)ds\Big). \tag{3.6}$$

This is the price of the defaultable zero-coupon bond given that it has not defaulted before time t. It then follows that the price of the defaultable bond can be written as

$$P^d(t,T) = \mathcal{R}(t)\bar{P}^d(t,T,\omega). \tag{3.7}$$

3.2 Embedding Stochastic Volatility within Defaultable HJM framework

Although the volatility processes in the standard HJM framework are path dependent, they are not considered to be stochastic in the sense of Hull and White (1987), Heston (1993) and Scott (1997). In a stochastic volatility model for interest rates, the volatility processes should be driven by Wiener processes which are independent of the Wiener processes driving the term structure of interest rates.

Chiarella and Kwon (1999) incorporated stochastic volatility within a class of HJM term structure models in the default-free setup and derived bond and bond option prices. Chiarella and Kwon (2001) adapted the Hobson and Rogers (1998) complete stochastic volatility stock market model to the interest rate setting and showed how the stochastic dynamics can be reduced to a Markovian form. This allowed for bond prices to be expressed in terms of the underlying state variables.

We assume that $f(t,T,\omega)$ and $\lambda(t,T,\omega)$ are the unique strong solutions to the stochastic integral equations

$$f(t,T,\omega) = f(0,T) + \int_0^t \alpha^f(u,T,\omega)du + \int_0^t \sigma^f(u,T,\omega)dW^f(u), \tag{3.8a}$$

$$\lambda(t,T,\omega) = \lambda(0,T) + \int_0^t \alpha^\lambda(u,T,\omega)du + \int_0^t \sigma^\lambda(u,T,\omega)dW^\lambda(u), \tag{3.8b}$$

or equivalently the stochastic differential equations,

$$df(t,T,\omega) = \alpha^f(t,T,\omega)dt + \sigma^f(t,T,\omega)dW^f(t), \tag{3.9a}$$

$$d\lambda(t,T,\omega) = \alpha^\lambda(t,T,\omega)dt + \sigma^\lambda(t,T,\omega)dW^\lambda(t), \tag{3.9b}$$

respectively. Schönbucher (1998) showed that a model of the spread for the defaultable interest rates over the default free interest rates may be used to add a default-risk module to an existing model of default-free interest rates. The forward credit spread therefore offers the link between the defaultable and default free term structures.

It follows from equations (3.8a) and (3.8b) that the stochastic integral equations for the instantaneous default-free short rate $r(t,\omega) := f(t,t,\omega)$ and the instantaneous short-term credit spread $\lambda(t,\omega) := \lambda(t,t,\omega)$ are given by

$$r(t,\omega) = f(0,t) + \int_0^t \alpha^f(u,t,\omega)du + \int_0^t \sigma^f(u,t,\omega)dW^f(u), \tag{3.10a}$$

$$\lambda(t,\omega) = \lambda(0,t) + \int_0^t \alpha^\lambda(u,t,\omega)du + \int_0^t \sigma^\lambda(u,t,\omega)dW^\lambda(u), \tag{3.10b}$$

respectively.

We assume that ω is the same under both $\sigma^f(t,T,\omega)$ and $\sigma^\lambda(t,T,\omega)$ and it is driven by the stochastic process V whose dynamics follow the stochastic differential equation

$$dV(t) = \alpha^V(V,t)dt + \sigma^V(V,t)dW^V(t), \tag{3.11}$$

where the drift and diffusion depend only on V. Examples of the volatility functions that we shall use are

$$\sigma^f(t,T,\omega) = \sigma_0^f(V(t))^{\gamma_1}(r(t,\omega))^{\gamma_2}e^{-\kappa_f(T-t)},$$

$$\sigma^\lambda(t,T,\omega) = \sigma_0^\lambda(V(t))^{\gamma_1}(\lambda(t,\omega))^{\gamma_3}e^{-\kappa_\lambda(T-t)},$$

and

$$\sigma^V(V,t) = \bar{\sigma}^v V(t)^{\gamma_1},$$

with $\gamma_i \geq 0, \quad i = 1,2,3$ and σ_0^f, σ_0^λ and $\bar{\sigma}^v$ which are constants.

By using equation (3.5), (3.8a) and (3.8b), the stochastic integral equation for the defaultable forward rate is expressed as

$$f^d(t,T,\omega) = f^d(0,T) + \int_0^t \alpha^d(u,T,\omega)du + \int_0^t \sigma^f(u,T,\omega)dW^f(u) + \int_0^t \sigma^\lambda(u,T,\omega)dW^\lambda(u), \tag{3.12}$$

where the initial defaultable forward curve is specified by

$$f^d(0,T) = f(0,T) + \lambda(0,T), \tag{3.13}$$

and the drift coefficient is given by the sum of the individual drift coefficients

$$\alpha^d(t,T,\omega) = \alpha^f(t,T,\omega) + \alpha^\lambda(t,T,\omega). \tag{3.14}$$

Then the instantaneous dynamics for the defaultable short rate $r^d(t,\omega) \equiv f^d(t,t,\omega)$ are given by

$$r^d(t,\omega) = f^d(0,t) + \int_0^t \alpha^d(u,t,\omega)du + \int_0^t \sigma^f(u,t,\omega)dW^f(u) + \int_0^t \sigma^\lambda(u,t,\omega)dW^\lambda(u). \tag{3.15}$$

3.3 Correlation Structure

Evidence of the effects of correlation between stochastic volatility and short rate on the bond price were investigated in Heston (1993). Jarrow and Turnbull (2000) showed that the correlation between the short rate and the credit spread represents the correlation empirically observed between market risk and credit risk. Changes in the default-free short rate compel investors to reassess the probability of default of the defaultable bonds and therefore change the credit spreads.

We define the correlation matrix between the Wiener processes $W^f(t)$, $W^\lambda(t)$ and $W^V(t)$ by

$$\mathbb{E}\big[(dW^V, dW^\lambda, dW^f)^\top(dW^V, dW^\lambda, dW^f)\big] = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{21} & 1 & \rho_{23} \\ \rho_{31} & \rho_{32} & 1 \end{bmatrix}, \tag{3.16}$$

where the correlation coefficients $\rho_{ij}\prime s$ are given by

$$\rho_{12}dt = \mathbb{E}[dW^V(t) \cdot dW^\lambda(t)], \tag{3.17a}$$

$$\rho_{13}dt = \mathbb{E}[dW^V(t) \cdot dW^f(t)], \tag{3.17b}$$

$$\rho_{23}dt = \mathbb{E}[dW^\lambda(t) \cdot dW^f(t)]. \tag{3.17c}$$

To apply the techniques of the HJM approach, it is convenient to replace the correlated Wiener processes $W^f(t)$, $W^\lambda(t)$ and $W^V(t)$ with uncorrelated processes. We define the uncorrelated Wiener process $W(t) = \big(W_1(t), W_2(t), W_3(t)\big)$ under $\mathbb{P}$ such that

$$\begin{bmatrix} dW^V(t) \\ dW^\lambda(t) \\ dW^f(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} dW_1(t) \\ dW_2(t) \\ dW_3(t) \end{bmatrix}. \tag{3.18}$$

Note that the a_{ij}'s are chosen such that the correlation structure of the Wiener processes $W^f(t)$, $W^\lambda(t)$ and $W^V(t)$ is preserved with

$$\sum_{k=1}^{3} a_{ik}a_{jk} = \rho_{ij}, \quad \text{for} \quad i \neq j, \quad j = 1,2,3, \quad \text{and} \quad \sum_{j=1}^{3} a_{ij}^2 = 1, \quad \text{for} \quad i = 1,2,3. \tag{3.19}$$

For instance, a typical choice would be $a_{12} = a_{13} = a_{23} = 0$, yieding the transformation

$$\begin{bmatrix} dW^V(t) \\ dW^\lambda(t) \\ dW^f(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \rho_{12} & \sqrt{1-\rho_{12}^2} & 0 \\ \rho_{13} & \frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}} & \sqrt{\frac{1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23}}{1-\rho_{12}^2}} \end{bmatrix} \begin{bmatrix} dW_1(t) \\ dW_2(t) \\ dW_3(t) \end{bmatrix}. \tag{3.20}$$

This is the particular transformation that we use in later numerical examples in Section **??**.

Then, equations (3.9a), (3.9b) and (3.11) can be rewritten as

$$df(t,T,\omega) = \alpha^f(t,T,\omega)dt + \sum_{i=1}^{3} \tilde{\sigma}_i^f(t,T,\omega)dW_i(t), \tag{3.21a}$$

$$d\lambda(t,T,\omega) = \alpha^\lambda(t,T,\omega)dt + \sum_{i=1}^{3} \tilde{\sigma}_i^\lambda(t,T,\omega)dW_i(t), \tag{3.21b}$$

$$dV(t) = \alpha^V(V,t)dt + \sum_{i=1}^{3} \tilde{\sigma}_i^V(t,T,\omega)dW_i(t), \tag{3.21c}$$

where using equations (3.18) and (3.20) we define the following volatility functions

$$\tilde{\sigma}_i^f(t,T,\omega) = a_{3i}\sigma^f(t,T,\omega), \quad \tilde{\sigma}_i^\lambda(t,T,\omega) = a_{2i}\sigma^\lambda(t,T,\omega), \quad \tilde{\sigma}_i^V(t,T,\omega) = a_{1i}\sigma^V(V,t), \tag{3.22}$$

for $i = 1, 2, 3$. Equation (3.21a) represents a multi-factor HJM model with three noise processes.

Using transformation (3.18), we can rewrite equation (3.12) as

$$f^d(t,T,\omega) = f^d(0,T) + \int_0^t \alpha^d(u,T,\omega)du + \sum_{i=1}^3 \int_0^t \tilde{\sigma}_i^d(u,T,\omega)dW_i(u), \tag{3.23}$$

where the volatility function is defined by

$$\tilde{\sigma}_i^d(t,T,\omega) = \tilde{\sigma}_i^f(t,T,\omega) + \tilde{\sigma}_i^\lambda(t,T,\omega). \tag{3.24}$$

The random loss $q(\tau_i)$ is considered as a random draw at default time τ_i. The fractional recovery process $\mathcal{R}(t)$ can be represented as a Doleans-Dade exponential of the stochastic differential equation [1]

$$d\mathcal{R}(t) = -\mathcal{R}(t-)q(t)dN(t). \tag{3.25}$$

We then have the following result for defaultable bond price dynamics.

Proposition 3.1 *The defaultable bond defined by equation* (3.4) *satisfies the stochastic differential equation*

$$\frac{dP^d(t,T)}{P^d(t-,T,\omega)} = \left[r^d(t,\omega) + b^d(t,T,\omega)\right]dt + \sum_{i=1}^3 \tilde{\sigma}_{B,i}^d(t,T,\omega)dW_i(t) - q(t)dN(t), \tag{3.26}$$

where

$$b^d(t,T,\omega) = -\alpha_B^d(t,T,\omega) + \frac{1}{2}\sum_{i=1}^3 \left(\tilde{\sigma}_{B,i}^d(t,T,\omega)\right)^2, \tag{3.27}$$

and

$$\alpha_B^d(t,T,\omega) = \int_t^T \alpha^d(t,s,\omega)ds, \quad \tilde{\sigma}_{B,i}^d(t,T,\omega) = -\int_t^T \tilde{\sigma}_i^d(t,s,\omega)ds.$$

Proof: See Appendix A. ■

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[1] For the details of proof to this key result, the reader is referred to Jacod and Shiryaev (2003), Theorem 4.61, pg.59.

3.4 Risk-Neutral Dynamics

The absence of arbitrage opportunities implies that there exists an equivalent probability measure $\tilde{\mathbb{P}}$, namely the risk-neutral measure. For every finite maturity T there exist a 3-dimensional predictable process $\Phi(t) = \{\phi_1(t), \phi_2(t), \phi_3(t), t \in [0,T]\}$ and a strictly positive measurable function $\psi(t)$ satisfying the integrability conditions

$$\int_0^t ||\phi_i(s)||^2 ds < \infty, \quad \text{for} \quad i = 1,2,3, \qquad \int_0^t |\psi(s)h(s)| ds < \infty, \tag{3.28}$$

such that

$$d\tilde{W}_i(t) = dW_i(t) - \phi_i(t)dt, \quad \text{for} \quad i = 1,2,3, \tag{3.29}$$

is a $\tilde{\mathbb{P}}$-Wiener process and the default indicator process $N(t)$ has a $\tilde{\mathbb{P}}$-intensity

$$\tilde{h}(t) = \psi(t)h(t). \tag{3.30}$$

Proposition 3.2 *Let the defaultable bond price follow the dynamics as given in Proposition 3.1*

$$\frac{dP^d(t,T)}{P^d(t-,T,\omega)} = \big[r^d(t,\omega) + b^d(t,T,\omega)\big]dt + \sum_{i=1}^{3} \tilde{\sigma}^d_{B,i}(t,T,\omega)dW_i(t) - q(t)dN(t). \tag{3.31}$$

Using Girsanov's theorem such that the integrability condition (3.28) *and equations* (3.29),(3.30) *are satisfied, then a risk-neutral measure* $\tilde{\mathbb{P}}$ *exists if and only if*

$$[r^d(t,\omega) + b^d(t,T,\omega)] + \sum_{i=1}^{3} \phi_i(t)\tilde{\sigma}^d_{B,i}(t,T,\omega) - \psi(t)q(t)h(t) = r(t,\omega). \tag{3.32}$$

Proof: See Appendix B. ■

Taking the derivative of (3.32) with respect to T and performing some standard manipulations then yields

$$\alpha^d(t,T,\omega) = -\sum_{i=1}^{3} \tilde{\sigma}^d_i(t,T,\omega)\Big(\phi_i(t) - \int_t^T \tilde{\sigma}^d_i(t,s,\omega)ds\Big), \tag{3.33}$$

which is the corresponding HJM forward rate drift restriction condition for the defaultable bond price. As noted in Schönbucher (2000), the precise knowledge of the nature of the default process N and its compensator M is not necessary in setting up an arbitrage free model for the term structure of defaultable bonds.

Substituting $b^d(t,T,\omega)$ as given in equation (3.27) into equation (3.32), as well as using (3.33), it follows that the short term spread is the product between the market price of jump risk, the default intensity and the expected loss quota,

$$r^d(t,\omega) - r(t,\omega) = \psi(t)q(t)h(t). \tag{3.34}$$

Taking into account the fact that the intensity of the default process under the risk-neutral measure is given by (3.30) then

$$r^d(t,\omega) - r(t,\omega) = q(t)\tilde{h}(t). \tag{3.35}$$

By using condition (3.32), equation (3.31) in Proposition 3.1 yields

$$\frac{dP^d(t,T)}{P^d(t-,T,\omega)} = r(t,\omega)dt + \sum_{i=1}^{3} \tilde{\sigma}^d_{B,i}(t,T,\omega)\big(dW_i(t) - \phi_i(t)dt\big) - q(t)\big(dN(t) - \psi(t)h(t)dt\big), \tag{3.36}$$

or from (3.29) and (3.30)

$$\frac{dP^d(t,T)}{P^d(t-,T,\omega)} = r(t,\omega)dt + \sum_{i=1}^{3} \tilde{\sigma}^d_{B,i}(t,T,\omega)d\tilde{W}_i(t) - q(t)d\tilde{M}(t). \tag{3.37}$$

These are the dynamics of the defaultable bond price under the risk-neutral measure in which $d\tilde{M}(t) = dN(t) - \tilde{h}(t)dt$ is a martingale.

We define the relative defaultable bond price by

$$Z^d(t,T,\omega) = \frac{P^d(t,T)}{B(t,\omega)},$$

where we recall that $B(t,\omega) = \exp\Big(\int_0^t r(s,\omega)ds\Big)$ is the accumulated money market account defined in equation (2.15). Applying Itô's quotient rule, the stochastic differential equation for $Z^d(t,T,\omega)$ is

$$\frac{dZ^d(t,T,\omega)}{Z^d(t-,T,\omega)} = \sum_{i=1}^{3} \tilde{\sigma}^d_{B,i}(t,T,\omega)d\tilde{W}_i(t) - q(t)d\tilde{M}(t). \tag{3.38}$$

If we let $\tilde{\mathbb{E}}$ denote mathematical expectation with respect to the risk neutral probability measure, it then follows that

$$\tilde{\mathbb{E}}\big[dZ^d(t,T,\omega)\big|\mathcal{F}_t\big] = 0.$$

This implies that

$$\tilde{\mathbb{E}}\big[Z^d(T,T,\omega)\big|\mathcal{F}_t\big] = Z^d(t,T,\omega),$$

and given that $P^d(T,T) = \bar{P}^d(T,T,\omega)\mathcal{R}(T)$, the defaultable bond price satisfies

$$P^d(t,T) = \tilde{\mathbb{E}}\Big[\exp\Big(-\int_t^T r(s,\omega)ds\Big)\mathcal{R}(T)\Big|\mathcal{F}_t\Big], \tag{3.39}$$

which on following the approach in Lando (1998) reduces to

$$P^d(t,T) = \mathcal{R}(t)\tilde{\mathbb{E}}\Big[\exp\Big(-\int_t^T \big(r(s,\omega)+\tilde{h}(s)q(s)\big)ds\Big)\Big|\mathcal{F}_t^N\Big], \tag{3.40}$$

where $\mathcal{F}_t^N$ is defined in (2.2). Note that the quantity $\exp\big(-\int_t^T r(s,\omega)ds\big)$ is the stochastic discount factor under measure $\tilde{\mathbb{P}}$ used to discount back to time t the \$1 payoff to be received at time T.

By using equation (3.24), the drift restriction condition (3.33) can be expanded to

$$\begin{aligned}\alpha^d(t,T,\omega) = &-\sum_{i=1}^3 \phi_i(t)\tilde{\sigma}_i^f(t,T,\omega) - \sum_{i=1}^3 \phi_i(t)\tilde{\sigma}_i^\lambda(t,T,\omega)\\ &+\sum_{i=1}^3 \tilde{\sigma}_i^f(t,T,\omega)\int_t^T \tilde{\sigma}_i^f(t,s,\omega)ds + \sum_{i=1}^3 \tilde{\sigma}_i^\lambda(t,T,\omega)\int_t^T \tilde{\sigma}_i^\lambda(t,s,\omega)ds\\ &+\sum_{i=1}^3 \tilde{\sigma}_i^\lambda(t,T,\omega)\int_t^T \tilde{\sigma}_i^f(t,s,\omega)ds + \sum_{i=1}^3 \tilde{\sigma}_i^f(t,T,\omega)\int_t^T \tilde{\sigma}_i^\lambda(t,s,\omega)ds.\end{aligned} \tag{3.41}$$

Then by using equation (3.23), (3.29) and the drift restriction condition (3.41), the defaultable forward rate risk-neutral dynamics are

$$\begin{aligned}f^d(t,T,\omega) = f^d(0,T) &+ \sum_{i=1}^3 \int_0^t \tilde{\sigma}_i^f(u,T,\omega)\int_u^T \tilde{\sigma}_i^f(u,s,\omega)dsdu\\ &+\sum_{i=1}^3 \int_0^t \tilde{\sigma}_i^\lambda(u,T,\omega)\int_u^T \tilde{\sigma}_i^\lambda(u,s,\omega)dsdu + \sum_{i=1}^3 \int_0^t \tilde{\sigma}_i^\lambda(u,T,\omega)\int_u^T \tilde{\sigma}_i^f(u,s,\omega)dsdu\\ &+\sum_{i=1}^3 \int_0^t \tilde{\sigma}_i^f(u,T,\omega)\int_u^T \tilde{\sigma}_i^\lambda(u,s,\omega)dsdu\\ &+\sum_{i=1}^3 \int_0^t \tilde{\sigma}_i^f(u,T,\omega)d\tilde{W}_i(u) + \sum_{i=1}^3 \int_0^t \tilde{\sigma}_i^\lambda(u,T,\omega)d\tilde{W}_i(u).\end{aligned} \tag{3.42}$$

We recall from Heath et al. (1992) that the default-free forward rate drift restriction condition in the multi-factor case is given by condition 2.7. By substituting (3.14) and

(3.24) into equation (3.33) and using condition 2.7 for $n = 3$, we obtain

$$\begin{aligned}\alpha^{\lambda}(t,T,\omega) = &-\sum_{i=1}^{3}\phi_i(t)\tilde{\sigma}_i^{\lambda}(t,T,\omega) + \sum_{i=1}^{3}\tilde{\sigma}_i^{\lambda}(t,T,\omega)\int_t^T \tilde{\sigma}_i^{\lambda}(t,s,\omega)ds \\ &+ \sum_{i=1}^{3}\Big(\tilde{\sigma}_i^{\lambda}(t,T,\omega)\int_t^T \tilde{\sigma}_i^{f}(t,s,\omega)ds + \tilde{\sigma}_i^{f}(t,T,\omega)\int_t^T \tilde{\sigma}_i^{\lambda}(t,s,\omega)ds\Big). \quad (3.43)\end{aligned}$$

Therefore the proposed model for the defaultable forward rate implies condition (3.43) for the credit spread drift[2]. The drift of the spread is given in terms of the volatilities of default-free forward rates and credit spread. This structure also guarantees that the spread cannot become negative.

Similarly, from condition (3.43) the forward credit spread dynamics $\lambda(t,T,\omega)$ in equation (3.21b) can be written as

$$\begin{aligned}\lambda(t,T,\omega) = \lambda(0,T) &+ \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^{\lambda}(u,T,\omega)\int_u^T \tilde{\sigma}_i^{\lambda}(u,s,\omega)dsdu \\ &+ \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^{f}(u,T,\omega)\int_u^T \tilde{\sigma}_i^{\lambda}(u,s,\omega)dsdu + \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^{\lambda}(u,T,\omega)\int_u^T \tilde{\sigma}_i^{f}(u,s,\omega)dsdu \\ &+ \sum_{i=1}^{3}\tilde{\sigma}_i^{\lambda}(t,t,\omega)d\tilde{W}_i(t). \quad (3.44)\end{aligned}$$

It follows from equation (3.42) that the instantaneous defaultable short rate dynamics $r^d(t,\omega)$ under the risk neutral measure are given by the stochastic integral equation

$$\begin{aligned}r^d(t,\omega) = f^d(0,t) &+ \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^{f}(u,t,\omega)\int_u^t \tilde{\sigma}_i^{f}(u,s,\omega)dsdu + \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^{\lambda}(u,t,\omega)\int_u^t \tilde{\sigma}_i^{\lambda}(u,s,\omega)dsdu \\ &+ \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^{\lambda}(u,t,\omega)\int_u^t \tilde{\sigma}_i^{f}(u,s,\omega)dsdu + \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^{f}(u,t,\omega)\int_u^t \tilde{\sigma}_i^{\lambda}(u,s,\omega)dsdu \\ &+ \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^{f}(u,t,\omega)d\tilde{W}_i(u) + \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^{\lambda}(u,t,\omega)d\tilde{W}_i(u). \quad (3.45)\end{aligned}$$

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[2] See Schönbucher (1998), corollary 1, equation (54) and in Pugachevsky (1999), equation (35).

In order to alleviate notational complexity, we define the path dependent quantities $S_j(t,\omega)$ and $\psi_j(t,\omega)$ such that:

$$S_1(t,\omega) = \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^f(u,t,\omega) \int_u^t \tilde{\sigma}_i^f(u,v,\omega) dv du, \tag{3.46a}$$

$$S_2(t,\omega) = \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^\lambda(u,t,\omega) \int_u^t \tilde{\sigma}_i^\lambda(u,v,\omega) dv du, \tag{3.46b}$$

$$S_3(t,\omega) = \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^f(u,t,\omega) \int_u^t \tilde{\sigma}_i^\lambda(u,v,\omega) dv du, \tag{3.46c}$$

$$S_4(t,\omega) = \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^\lambda(u,t,\omega) \int_u^t \tilde{\sigma}_i^f(u,v,\omega) dv du, \tag{3.46d}$$

$$\psi_1(t,\omega) = \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^f(u,t,\omega) d\tilde{W}_i(u), \tag{3.46e}$$

$$\psi_2(t,\omega) = \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^\lambda(u,t,\omega) d\tilde{W}_i(u). \tag{3.46f}$$

Then, the defaultable short rate in equation (3.45) satisfies the stochastic integral equation

$$r^d(t,\omega) = f^d(0,t) + \sum_{j=1}^{4} S_j(t,\omega) + \sum_{j=1}^{2} \psi_j(t,\omega). \tag{3.47}$$

This can be written in differential form as

$$dr^d(t,\omega) = \left[f_2^d(0,t) + \sum_{j=1}^{4} \frac{\partial}{\partial t} S_j(t,\omega)\right] dt + \sum_{j=1}^{2} d\psi_j(t,\omega). \tag{3.48}$$

From Definition 3.5, we have $\lambda(t,\omega) =: \lambda(t,t,\omega) = r^d(t,\omega) - r(t,\omega)$. To model the short term credit spread, $\lambda(t,\omega) =: \lambda(t,t,\omega)$ under fractional recovery, we use equation (3.35) to obtain

$$\lambda(t,\omega) \equiv \tilde{h}(t) q(t). \tag{3.49}$$

Formulating the intensity rate as a stochastic process allows rich dynamics for the credit spread process and is flexible enough to capture the empirically observed stochastic credit spreads. Using equation (3.44) and definition 3.5, its dynamics are modelled via the stochastic integral equation

$$\lambda(t,\omega) = \lambda(0,t) + \sum_{j=2}^{4} S_j(t,\omega) + \psi_2(t,\omega). \tag{3.50}$$

As cited in Jarrow and Turnbull (2000), there is considerable empirical evidence that clearly suggests that the credit spread is a function of at least default intensity and the recovery process. Pan and Singleton (2008) further noted that since $\tilde{h}(t)$ and $q(t)$ enter symmetrically into pricing under fractional recovery of market value(RMV), they cannot be separately identified using defaultable bond price data alone. However, for CDS pricing where the fractional recovery of face value(RFV) pricing framework due to Duffie (1998) and Duffie and Singleton (1999) is more natural, these two play distinct roles.

Jarrow and Turnbull (2000) also suggested that the default intensity process could be assumed to depend on different state variables to reflect a dependency on several macro-economic factors. This requirement could be well captured by a multi-factor model of the type in equation (3.50). The stochastic volatility process $V(t)$ in equation (3.21c) under the risk-neutral measure $\tilde{\mathbb{P}}$ follows the stochastic differential equation

$$dV(t) = \big[\alpha^V(V,t) + \phi_1(t)\tilde{\sigma}_1^V(t,T,V)\big]dt + \tilde{\sigma}_1^V(t,T,V)d\tilde{W}_1(t). \tag{3.51}$$

The market price for the risk factor $W_1(t)$ appears in the drift of the volatility processes since volatility is not a tradeable factor and the dependence on V is explicitly denoted due to its path dependence.

Using the relationship (3.49), the bond price formula under the expectations operator in equation (3.40) could then be written as

$$P^d(t,T) = \mathcal{R}(t)\tilde{\mathbb{E}}\Big[\exp\Big(-\int_t^T r^d(s,\omega)ds\Big)\Big|\mathcal{F}_t^W\Big]. \tag{3.52}$$

At the simplest level, the expectation in equation (3.52) could be calculated numerically by simulating the stochastic differential equation (3.48) for $r^d(t,\omega)$, the recovery process $\mathcal{R}(t)$ in equation (3.25) and stochastic volatility process in equation (3.51).

4 Markovian Structure

A key drawback of the HJM approach is the non-Markovian nature of the spot interest rates in its most general form, thereby increasing the computational complexity. This feature also makes it difficult to obtain expressions for prices of term-structure contingent claims in terms of partial differential equations. In this case, the prices are expressed as expectation operators of the payoffs under the equivalent martingale measure.

The subsidiary state variables play a central role in allowing us to transform the original non-Markovian dynamics to Markovian form. These variables have been used in default-free framework by Cheyette (1992), Ritchken and Sankarasubramanian (1995), Inui and Kijima (1998), Bhar and Chiarella (1997), Björk and Svensson (2001), Björk and Landèn (2002) and Chiarella and Kwon (2001a) to summarize the history of the path of the forward rate volatility. A consequence of this Markovianization process is that it leads to a generalized $(n+1)$ dimensional Markovian dynamical system.

4.1 The Ritchken-Sankarasubramanian Model with Stochastic Volatility

The Ritchken and Sankarasubramanian (1995) interest rate model (hereafter referred to as RS model) is characterized by the separability of the volatility structure of instantaneous forward rates permitting a two-state variable representation. This separation of the volatility structure was first discussed in Jamshidian (1991) who called the class *Quasi Gaussian.* In particular, in the RS model they show that if the volatility has the form

$$\sigma^f(t,T) = \sigma^r(t)k(t,T), \tag{4.1}$$

where $\sigma^r(t) = \sigma^f(t,t)$, the volatility of the short rate $r(t)$ is a function that depends on all information up to time t, and

$$k(t,T) = \exp\Big(-\int_t^T \kappa(x)dx\Big),$$

is a deterministic function satisfying the semi-group property

$$k(t,T) = k(t,u)k(u,T), \quad k(t,t) = 1 \quad \forall t \in [u,T].$$

Then, conditional on knowing the initial term structure, the knowledge of any two points on the term structure at time t is sufficient to characterize the full yield curve at that time.

Chiarella and Kwon (2001a) generalized the equation for the bond price to forward rate dependent volatility processes. By setting $T = t+\tau$, the stochastic integral equation of the forward rate may also be written in the form more convenient for the LIBOR models, namely

$$f(t,t+\tau,\omega) = f(0,t+\tau) + \sum_{i=1}^{2}\int_0^t \tilde{\sigma}_i^{f*}(u,t+\tau,\omega)du + \sum_{i=1}^{2}\int_0^t \tilde{\sigma}_i^{f}(u,t+\tau,\omega)d\tilde{W}_i(u), \tag{4.2}$$

where $\tilde{\sigma}_i^{f*}(t,t+\tau,\omega) = \tilde{\sigma}_i^f(t,t+\tau,\omega)\int_t^{t+\tau}\tilde{\sigma}_i^{f*}(u,t+\tau,\omega)du$.

Chiarella and Kwon (1999) defined the subsidiary state variables for the model by

$$\psi(t,\tau) = \int_0^t \sigma^2(u,t+\tau)du, \tag{4.3a}$$

$$\zeta(t,\tau) = \sum_{i=1}^{2}\Big[\int_0^t \tilde{\sigma}_i^{f*}(u,t+\tau)du + \int_0^t \tilde{\sigma}_i^f(u,t+\tau)d\tilde{W}_i(u)\Big], \tag{4.3b}$$

with special cases when $\psi(t) = \psi(t,0)$ and $\zeta(t) = \zeta(t,0)$ where we have suppressed the dependence on ω for notational simplicity. It is then shown that the state variables

$$\psi(t,\tau) = \alpha^2(t,\tau)\psi(t), \tag{4.4a}$$

$$\zeta(t,\tau) = \alpha(t,\tau)\zeta(t) + \gamma(t,\tau)\psi(t), \tag{4.4b}$$

where $\alpha(t,\tau) = e^{-\int_t^{t+\tau}\kappa(u)du}$ and $\gamma(t,\tau) = \alpha(t,\tau)\int_t^{t+\tau}\alpha(t,u{-}t)du$, satisfy the equations

$$d\eta(t) = [\sigma^2(t,t) - 2\kappa(t)\eta(t)]dt, \tag{4.5a}$$

$$d\zeta(t) = [\eta(t) - \kappa(t)\zeta(t)]dt + \sigma^2(t,t)[\rho d\tilde{W}_1(t) + \sqrt{1-\rho^2}d\tilde{W}_2(t)], \tag{4.5b}$$

while the volatility function $V(t)$ follows the stochastic differential equation

$$dV(t) = \mu_v dt + \sigma_v d\tilde{W}_1(t). \tag{4.6}$$

Given that $\beta(t,T) = \int_t^T \alpha(t,u-t)du$, it is then shown that the bond price is given by

$$P(t,T) = \frac{P(0,T)}{P(0,t)}\exp\Big(-\beta(t,T)\zeta(t) - \frac{1}{2}\beta^2(t,T)\psi(t)\Big). \tag{4.7}$$

4.2 Defaultable HJM model under the RS framework

The non-Markovian noise term in the stochastic differential equation (3.48) for $r^d(t,\omega)$ would constitute a principal source of difficulty in implementing and estimating the HJM model. The components

$$\frac{\partial}{\partial t}\int_0^t \tilde{\sigma}_i^f(u,t,\omega)\int_u^t \tilde{\sigma}_i^f(u,s,\omega)dsdu, \quad \frac{\partial}{\partial t}\int_0^t \tilde{\sigma}_i^\lambda(u,t,\omega)\int_u^t \tilde{\sigma}_i^\lambda(u,s,\omega)dsdu,$$
$$\frac{\partial}{\partial t}\int_0^t \tilde{\sigma}_i^f(u,t,\omega)\int_u^t \tilde{\sigma}_i^\lambda(u,s,\omega)dsdu, \quad \frac{\partial}{\partial t}\int_0^t \tilde{\sigma}_i^\lambda(u,t,\omega)\int_u^t \tilde{\sigma}_i^f(u,s,\omega)dsdu,$$

$$\int_0^t \frac{\partial}{\partial t}\tilde{\sigma}_i^f(u,t,\omega)d\tilde{W}_i(u) \quad \text{and} \quad \int_0^t \frac{\partial}{\partial t}\tilde{\sigma}_i^\lambda(u,t,\omega)d\tilde{W}_i(u),$$

could depend on the path history of the noise process from time 0 to current time t.

We now consider a class of functional forms of volatility functions $\sigma^f(t,T,\omega)$ and $\sigma^\lambda(t,T,\omega)$ that allow the non-Markovian representation of $r^d(t,\omega)$ and $P^d(t,T)$ to be reduced to a finite dimensional Markovian system of stochastic differential equations.

We consider the case where the volatility function is a product of a deterministic function of time and a function of the short rate or intensity rate.

Assumption 4.1 *The volatility functions for the default-free forward interest rate and forward credit spread are of the form*

$$\sigma^f(t,T,\omega) = \bar{\sigma}_f V^\gamma(t) G_r(r(t,\omega)) e^{-\kappa_f(T-t)}, \tag{4.9a}$$

$$\sigma^\lambda(t,T,\omega) = \bar{\sigma}_\lambda V^\gamma(t) G_\lambda(\lambda(t,\omega)) e^{-\kappa_\lambda(T-t)}, \tag{4.9b}$$

respectively, with $\gamma = 0.5$ *and*

$$G_r(r(t,\omega)) = \sqrt{r(t,\omega)} \quad \textit{and} \quad G_\lambda(\lambda(t,\omega)) = \sqrt{\lambda(t,\omega)},$$

where $\bar{\sigma}_f \geq 0$, $\bar{\sigma}_\lambda \geq 0$, κ_f *and* κ_λ *are constants.*

In particular, for $\gamma \geq 0$ we note that

$$\sigma^f(t,t,\omega) = \bar{\sigma}_f V^\gamma(t) G_r(r(t,\omega)), \tag{4.10a}$$

$$\sigma^\lambda(t,t,\omega) = \bar{\sigma}_\lambda V^\gamma(t) G_\lambda(\lambda(t,\omega)). \tag{4.10b}$$

We then have the following proposition.

Proposition 4.1 *Under the volatility specification in Assumption 4.1, the defaultable short rate satisfies the Markovian system of stochastic differential equations*

$$\begin{aligned} dr^d(t,\omega) = \big[&\theta_d(t,\omega) + \eta_1(t,\omega) + \eta_2(t,\omega) + 2\eta_3(t,\omega) - (\kappa_f - \kappa_\lambda)S_3(t,\omega) + (\kappa_f - \kappa_\lambda)\lambda(t,\omega) \\ &- \kappa_f r^d(t,\omega)\big]dt + \Big(\sum_{i=1}^{3} a_{3i}\bar{\sigma}_f\sqrt{r(t,\omega)V(t)} + \sum_{i=1}^{3} a_{2i}\bar{\sigma}_\lambda\sqrt{\lambda(t,\omega)V(t)}\Big)d\tilde{W}_i(t), \end{aligned} \tag{4.11}$$

where the coefficient $\theta_d(t,\omega)$ *in the drift is given by*

$$\theta_d(t,\omega) = f_2^d(0,t) + \kappa_f f(0,t) + \kappa_\lambda \lambda(0,t),$$

and the state variables $\eta_1, \eta_2, \eta_3, S_3$ *satisfy the SDE's*

$$d\eta_1(t,\omega) = [\sum_{i=1}^{3} a_{3i}^2 \bar{\sigma}_f^2 r(t,\omega)V(t) - 2\kappa_f \eta_1(t,\omega)]dt, \tag{4.12}$$

$$d\eta_2(t,\omega) = [\sum_{i=1}^{3} a_{2i}^2 \bar{\sigma}_\lambda^2 \lambda(t,\omega)V(t) - 2\kappa_\lambda \eta_2(t,\omega)]dt, \tag{4.13}$$

$$d\eta_3(t,\omega) = [\sum_{i=1}^{3} a_{3i}a_{2i}\bar{\sigma}_f\bar{\sigma}_\lambda \sqrt{r(t,\omega)\lambda(t,\omega)}V(t) - (\kappa_f + \kappa_\lambda)\eta_3(t,\omega)]dt, \tag{4.14}$$

$$dS_3(t,\omega) = \Big[\eta_3(t,\omega) - \kappa_f S_3(t,\omega)\Big]dt. \tag{4.15}$$

From equations (2.12) *and* (3.50)*, the default-free short rate and intensity rate follow the SDE's*

$$dr(t,\omega) = \big[\theta_f(t,\omega) + \eta_1(t,\omega) - \kappa_f r(t,\omega)\big]dt + \sum_{i=1}^{3} a_{3i}\bar{\sigma}_f \sqrt{r(t,\omega)V(t)}d\tilde{W}_i(t), \tag{4.16}$$

$$\begin{aligned} d\lambda(t,\omega) = {} & \big[\theta_\lambda(t,\omega) + \eta_2(t,\omega) + 2\eta_3(t,\omega) - (\kappa_f - \kappa_\lambda)S_3(t,\omega) - \kappa_\lambda\lambda(t,\omega)\big]dt \\ & + \sum_{i=1}^{3} a_{2i}\bar{\sigma}_\lambda \sqrt{\lambda(t,\omega)V(t)}d\tilde{W}_i(t), \end{aligned} \tag{4.17}$$

where the functions in the drifts are given by

$$\begin{aligned} \theta_f(t,\omega) &= f_2(0,t) + \kappa_f f(0,t), \\ \theta_\lambda(t,\omega) &= \lambda_2(0,t) + \kappa_\lambda \lambda(0,t). \end{aligned}$$

Proof: The proof to this proposition is found in Appendix C. ■

The volatility functions for the stochastic volatility process are given by $\tilde{\sigma}_1^V(t,T,\omega) = \bar{\sigma}^V\sqrt{V(t)}$ and $\tilde{\sigma}_2^V(t,T,\omega) = \tilde{\sigma}_3^V(t,T,\omega) = 0$. Using equation (3.51), this follows the stochastic differential equation

$$dV(t) = \big[\kappa_V\big(\bar{V} - V(t)\big) + \bar{\sigma}^V\phi_1(t)\sqrt{V(t)}\big]dt + \bar{\sigma}^V\sqrt{V(t)}d\tilde{W}_1(t). \tag{4.18}$$

If we take the market price of risk to be given by $\phi_1(t) = \sqrt{V(t)}$, then we can rearrange equation (4.18) to yield

$$dV(t) = \big[\theta_V - \varpi V(t)\big]dt + \bar{\sigma}^V\sqrt{V(t)}d\tilde{W}_1(t), \tag{4.19}$$

where $\theta_V = \kappa_v\bar{V}$, $\varpi = \kappa_v - \bar{\sigma}^V$.

4.3 Bond Price as a Function of the State Variables

The one-dimensional bond price formula for HJM models driven by separable volatility functions was first obtained by Ritchken and Sankarasubramanian (1995). Inui and Kijima (1998) later extended the framework to the general case. In this section, we seek to derive the bond price formula following the approach used in Chiarella and Kwon (2000) and Chiarella and Kwon (2003).

We show that discount bond prices across all maturities can be expressed in terms of the Markovian short rate and other Markovian state variables. We define the functions

$$\beta_f(t,T) = \int_t^T e^{-\kappa_f(v-t)}dv \quad \text{and} \quad \beta_\lambda(t,T) = \int_t^T e^{-\kappa_\lambda(v-t)}dv. \tag{4.20}$$

Since the coefficients κ_f and κ_λ are deterministic, then $\beta_f(t,T)$ and $\beta_\lambda(t,T)$ are also deterministic.

The following theorem gives an analytical expression of the defaultable bond price.

Theorem 4.2 *Assume that the dynamics of the defaultable short rate as are given in Proposition 4.1. In addition, let the functions defined in equation* (4.20) *be deterministic. Then, within the RS framework the price of a defaultable bond is exponential affine and is given by*

$$P^d(t,T) = \frac{\bar{P}^d(0,T)}{\bar{P}^d(0,t)}\exp\Big(- D(t,T) - \beta_f(t,T)\zeta_f(t,\omega) - \beta_\lambda(t,T)\zeta_\lambda(t,\omega)\Big), \tag{4.21}$$

where we define

$$\zeta_f(t,\omega) = r(t,\omega) - f(0,t), \quad \zeta_\lambda(t,\omega) = \lambda(t,\omega) - \lambda(0,t),$$

and the coefficient $D(t,T,\omega)$ is given by and

$$\begin{aligned} D(t,T) = {} & -\ln\mathcal{R}(t) + \frac{1}{2}\beta_f^2(t,T)\eta_1(t,\omega) + \frac{1}{2}\beta_\lambda^2(t,T)\eta_2(t,\omega) + A(t,T)\eta_3(t,\omega) \\ & + \big[\beta_f(t,T) + \beta_\lambda(t,T)\big]S_3(t,\omega), \end{aligned}$$

with

$$A(t,T) = \frac{1}{\kappa_f}\beta_f(t,T) + \frac{1}{\kappa_\lambda}\beta_\lambda(t,T) + \Big(\frac{1}{\kappa_f} + \frac{1}{\kappa_\lambda}\Big)\Big(\frac{1}{\kappa_f + \kappa_\lambda}\Big)\Big(1 - e^{-(\kappa_f+\kappa_\lambda)(T-t)}\Big).$$

Proof: See Appendix D. ■

5 Numerical Results

In this section, we investigate the effect of varying the stochastic volatility, correlation, recovery and default intensity on the distribution of the defaultable bond price and defaultable bond returns. The simulation experiment was done using different sets of default intensity and recovery parameters while two ratings classes, Aa and Ba.

We recall that under recovery of market value (RMV), the recovery ratio is a fraction of the current market value (Duffie and Singleton (1997), Duffie and Huang (1996), among others). This offers greater computational tractability as compared to other recovery models. In some models, the recovery rate becomes interwoven with the risk premium, making the distinction between intensity(hazard) and recovery rate effectively irrelevant. Except for the scenario when we are going to vary the volatility of volatility $\bar{\sigma}^V$, we will use the set of parameters given in Table 1 and initial term structures of forward rate and forward credit spread given by $f(0,T) = 0.08 - 0.03e^{-1.5T}$ and $\lambda(0,T) = 0.02 - 0.01e^{-1.3T}$ respectively. We make a simplistic assumption that the initial credit spread remains the same irrespective of the rating class.

We begin the simulation analysis by showing the effects of the correlations and vol of vol on the pseudo-bond price and on the normalised returns as shown in figures 5.1, 5.2 and 5.3

T	$\bar{\sigma}^f$	$\bar{\sigma}^\lambda$	$\bar{\sigma}^V$	$\bar{V}$	κ_f	κ_λ	κ_v
1.0	0.02	0.04	0.10	0.0857	0.60	0.40	0.20

Table 1: The parameter values used in the simulation experiment.

Figure 5.1 shows the effects of increasing the correlation between the stochastic volatility process and the short-term credit spread. As documented in Table 2, it is observed that increasing this correlation increases the skewness of the distribution of both the pseudo-bond price and pseudo-bond returns thereby skewing the distributions to the right. However, the increment in the correlation reduces the kurtosis in both cases.

In Figure 5.3 and Table 3, we observe that increasing correlation between the short-term credit spread and the default-free short rate increases the kurtosis and skewness of both distributions.

From Figure 5.3 and Table 4, we observe that increasing the vol of vol reduces the skewness of the pseudo-bond price and bond returns. However, this increases the kurtosis of both the pseudo-bond price and the bond returns.

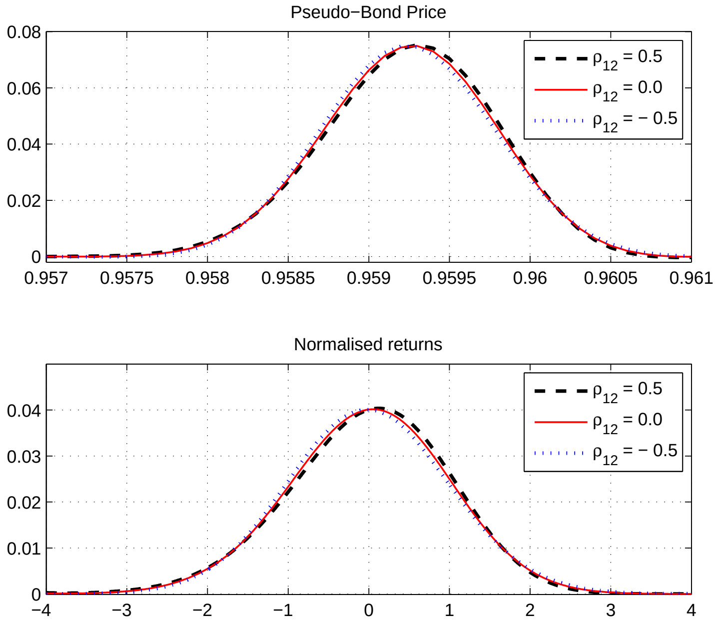

Figure 5.1: Distribution of pseudo-bond price under varying ρ_{12}.

ρ_{12}	Kurtosis(Price)	Skewness(Price)	Kurtosis(Returns)	Skewness(Returns)
-0.5	3.0872	-0.0479	3.0869	-0.0463
0.0	3.1113	-0.1071	3.1120	-0.1087
0.5	3.1778	-0.2627	3.1795	-0.2644

Table 2: An analysis of the effect of increasing the correlation between stochastic volatility and short-term credit spread, ρ_{12}, on the kurtosis and skewness of the pseudo-bond price and returns distribution given that the default intensity $\tilde{h}(t) = 0.0$.

ρ_{23}	Kurtosis(Price)	Skewness(Price)	Kurtosis(Returns)	Skewness(Returns)
-0.5	3.2606	-0.3982	3.2622	-0.3992
0.0	3.1976	-0.2998	3.1993	-0.3012
0.5	3.1743	-0.2559	3.1760	-0.2576

Table 3: An analysis of the effect of increasing the correlation between credit spread and short rate, ρ_{23}, on the kurtosis and skewness of the pseudo-bond price and returns distribution given that the default intensity $\tilde{h}(t) = 0.0$.

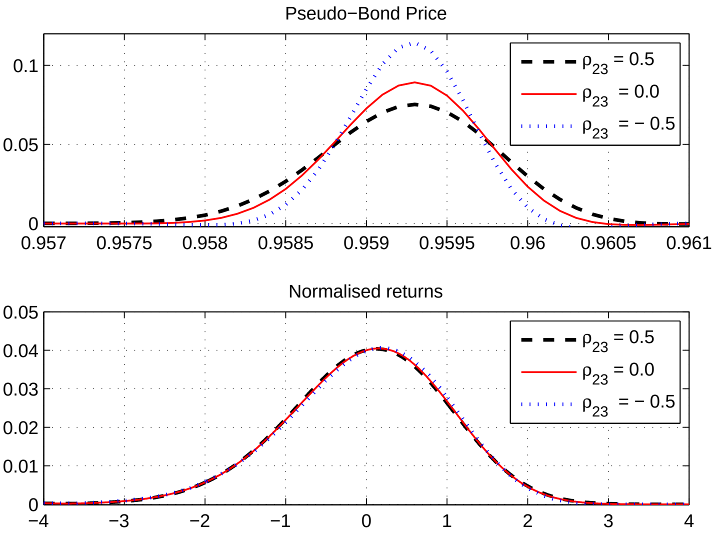

Figure 5.2: Distribution of pseudo-bond price under varying ρ_{23}.

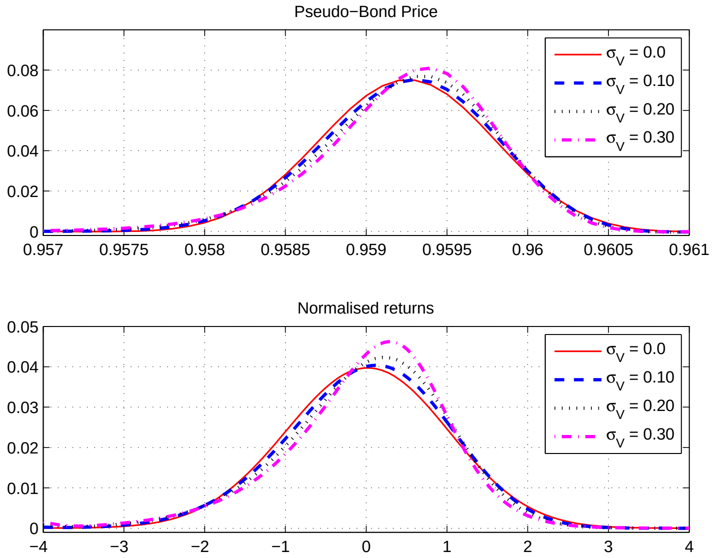

Figure 5.3: Distribution of pseudo-bond price under varying σ_V.

Vol of Vol σ^v	Kurtosis(Price)	Skewness(Price)	Kurtosis(Returns)	Skewness(Returns)
0.0	3.0010	-0.0377	3.0013	-0.0392
0.10	3.1776	-0.2627	3.1795	-0.2644
0.20	3.6439	-0.4876	3.6475	-0.4896
0.30	4.3989	-0.7116	4.4053	-0.7140

Table 4: An analysis on the effect of increasing $\bar{\sigma}^V$ on the kurtosis and skewness of the pseudo-bond price and bond returns distribution given default intensity $\tilde{h}(t) = 0.0$.

In addition to the parameter specifications used so far, we then proceeded to a more general setting where the bonds actually defaulted and there was subsequent recovery. The case of consideration under this was for bonds with default intensity $\tilde{h}(t) = 0.008$. This reflects the default intensity for Aa rated bonds and consequently we choose a recovery rate $R(t) = 54.44\%$ as documented by Moody's (1992 - 2003) report. We assumed that the loss given default, $LGD \sim \mathcal{N}(0.4556, 0.07)$.[3]

Figure 5.4 shows the histogram of the defaultable bond and normalised bond price using the parameters in Table 1 in addition to the respective default intensity and recovery rate. From the histograms, we observe that the prices cluster over two intervals exhibiting a bi-modal type of distribution. This is shown in figure 5.5 where we focus on the two intervals. In figure 5.6, we investigate the effects of correlations ρ_{12} and ρ_{23} on the distribution of defaultable bond price. From the numerical simulations, it is observed that these increases in the correlation do not affect interval 2, the left tail of the distribution. Interval 1 which captures the distribution of bonds that have not yet defaulted exhibits the same characteristics as the pseudo-bonds with regard to changes in the level of correlation.

Similarly, we observe that the defaultable bond returns also exhibit the same characteristics as the bond prices. The bond returns and normalised bond returns cluster in two intervals as shown in figure 5.8. On zooming-in on interval1 and 2 of the histograms, we observe that these exhibit a normal type of distribution which is consistent with the distribution of the recovery model used in the simulation experiment as shown in figure 5.9 - figure 5.11.

In addition, we observe that the skewness and kurtosis for the defaultable bond

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[3] Although our model allows for multiple defaults and recovery, we assume that the firm's default intensity and recovery rate remains the same even after default and restructuring. A more realistic specification would allow for downgrade in the credit quality thereby increasing the default intensity and reducing the recovery rate in the eventuality of future events.

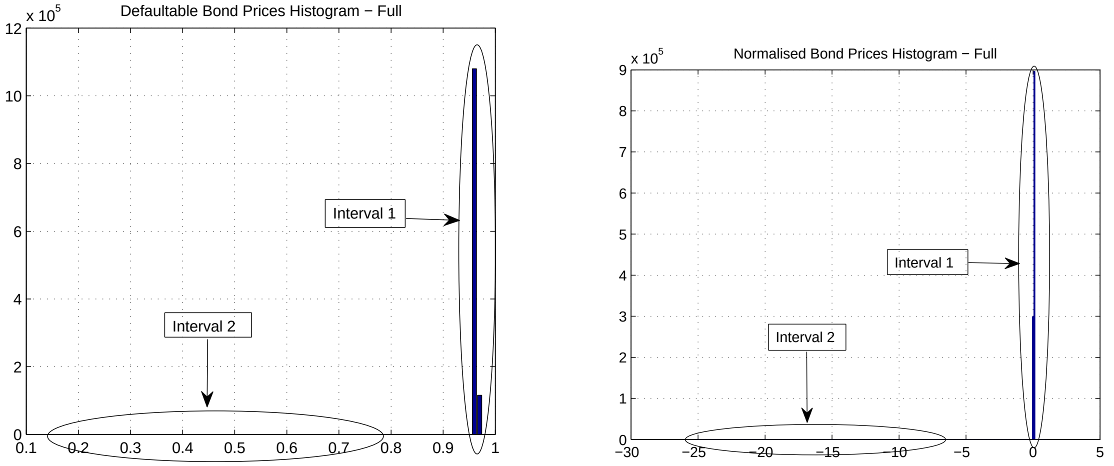

Figure 5.4: Histograms of defaultable and normalised bond price for Aa rated bonds.

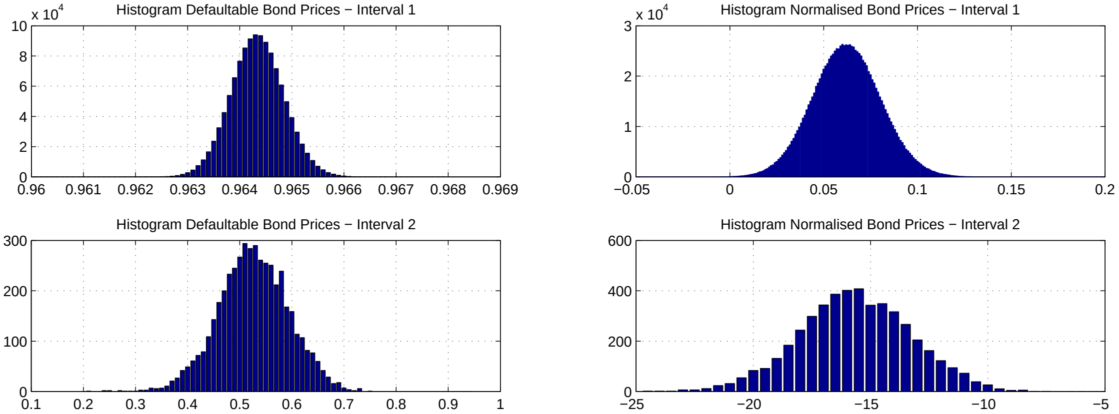

Figure 5.5: Zoomed-in histograms of defaultable and normalised bond price for Aa rated bonds.

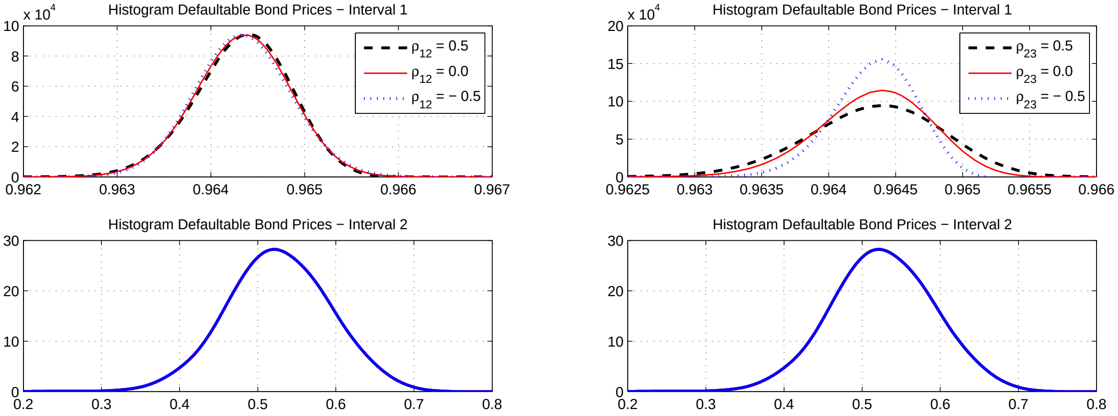

Figure 5.6: Distribution of defaultable bond price for Aa rated bonds under varying ρ_{12} and ρ_{23}.

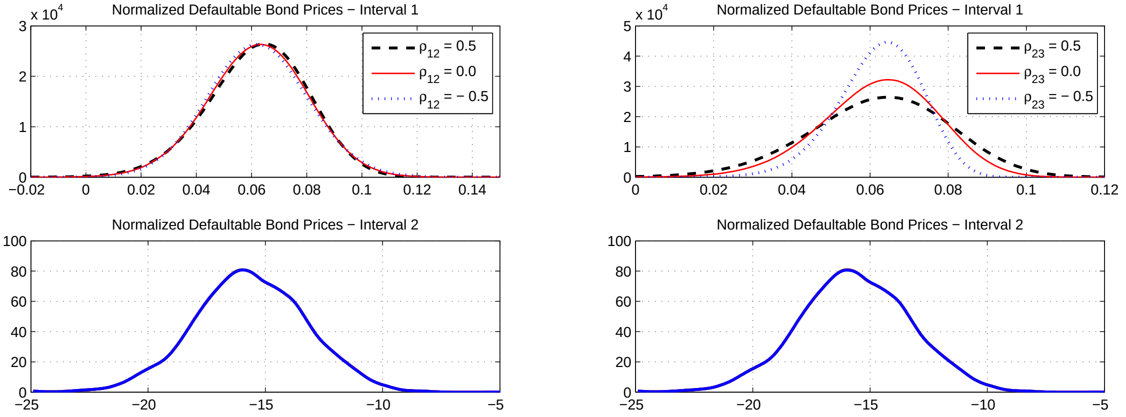

Figure 5.7: Distribution of normalised bond prices for Aa rated bonds under varying ρ_{12} and ρ_{23}.

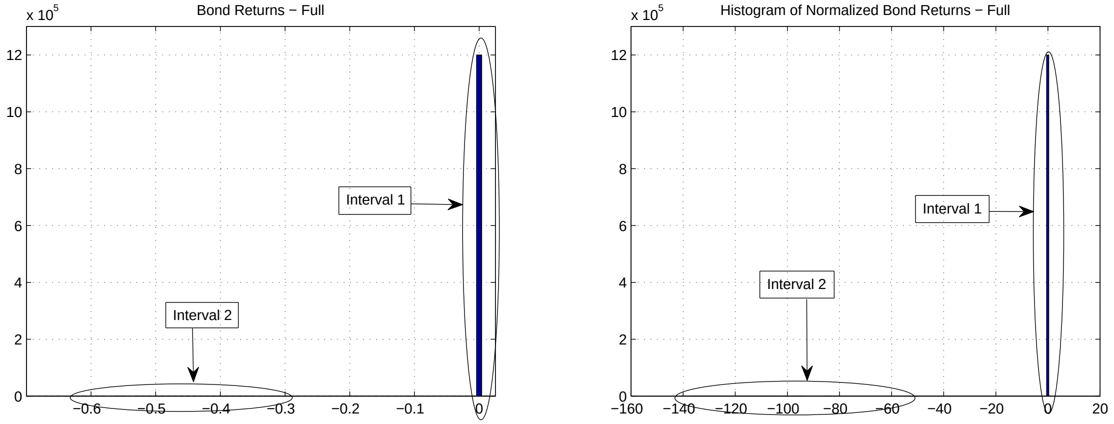

Figure 5.8: Histograms of defaultable and normalised bond returns for Aa rated bonds.

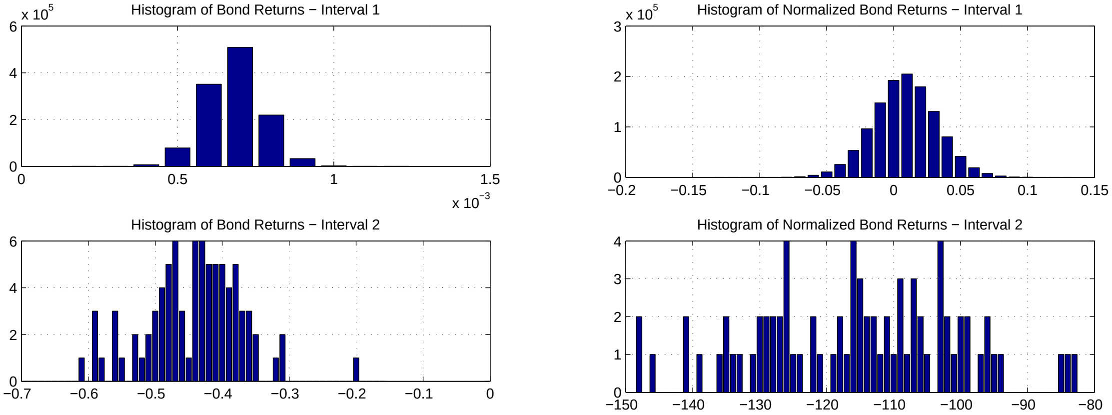

Figure 5.9: Zoomed histogram of defaultable and normalised bond returns for Aa rated bonds.

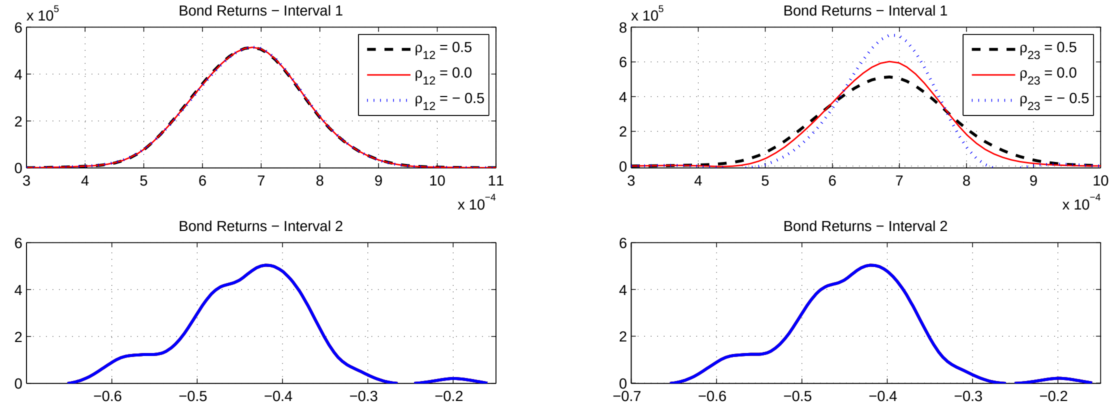

Figure 5.10: Distribution of defaultable bond returns for Aa rated bonds under varying ρ_{12} and ρ_{23}.

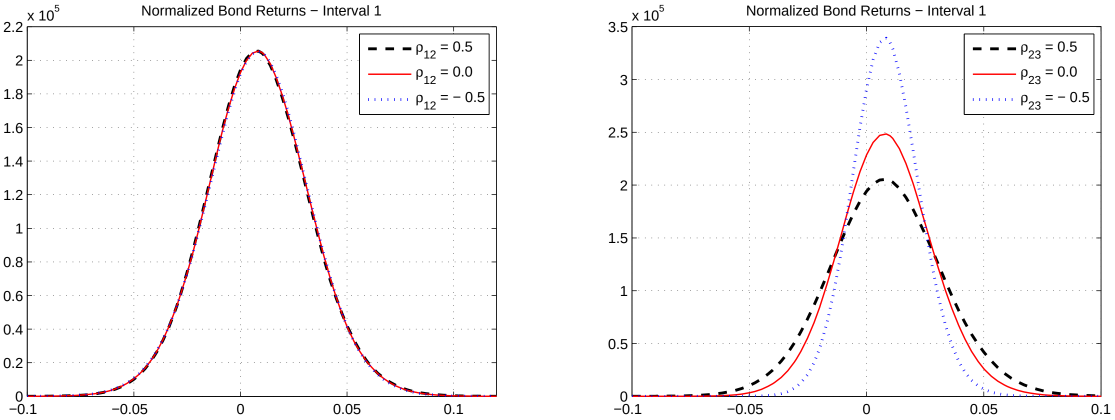

Figure 5.11: Distribution of normalised bond returns for Aa rated bonds under varying ρ_{12} and ρ_{23}.

price and bond returns are as given in Table 5. These values remain unchanged despite variation in the various correlations.

	Skewness	Kurtosis
Defaultable Bond Price	-16.5767	281.7459
Defaultable Bond Returns	-144.8717	2.1349×10^4

Table 5: Skewness and Kurtosis under defaultable bond dynamics for Aa rated bonds.

It was noted in D'Souza et al. (2004) that defaultable bonds have returns that are negatively skewed. This, they observed was due to the fact that the probability of defaultable bonds earning a substantial price appreciation is relatively small but there exists a large probability of earning small profit through interest rates earnings. The distribution, as observed also in our simulated results tends to be skewed around a positive value with a small positive tail reflecting the limited upside potential. Adverse movements in credit quality occur with small probability but these can have adverse negative impact on the value of the asset, generating significant losses. It has been empirically observed that skewed returns with heavy downside tails are characteristic of portfolios of defaultable bonds.

6 State Variables as functions of Forward Rates

In this section, we seek establish the economic interpretation of the state variables $\zeta_\lambda(t,\omega)$, $\zeta_\lambda(t,\omega)$, $S_3(t,\omega)$ and $\eta_i(t,\omega)$ for $i = 1,2,3$ and consequently obtain a connection between the defaultable bond price formula in equation (4.21) and market observable quantities.

We define the deterministic variables $\alpha_f(t,T)$ and $\alpha_\lambda(t,T)$ such that

$$\alpha_f(t,T) = e^{-\kappa_f(T-t)} \quad \text{and} \quad \alpha_\lambda(t,T) = e^{-\kappa_\lambda(T-t)}.$$

For $u \le t \le T$, the volatility functions $\sigma^f(t,T,\omega)$ and $\sigma^\lambda(t,T,\omega)$ satisfy the identities

$$\int_0^t \sigma^f(u,T,\omega)du = \alpha_f(t,T)\int_0^t \sigma^f(u,t,\omega)du, \tag{6.1a}$$

$$\int_0^t \sigma^\lambda(u,T,\omega)du = \alpha_\lambda(t,T)\int_0^t \sigma^\lambda(u,t,\omega)du. \tag{6.1b}$$

The notable feature of (6.1a) & (6.1b) is that this specific choice permits time and maturity dependence in the components of the forward rate volatility process to be separated.

Let $\Delta f^d(t,T,\omega) = f^d(t,T,\omega) - f^d(0,T)$. Then using equation (3.22), equation (3.42) can be written

$$\begin{aligned}\Delta f^d(t,T,\omega) &= \sum_{i=1}^{3} a_{3i}^2 \int_0^t \sigma^f(u,T) \int_u^T \sigma^f(u,s)dsdu + \sum_{i=1}^{3} a_{2i}^2 \int_0^t \sigma^\lambda(u,T) \int_u^T \sigma^\lambda(u,s)dsdu \\ &+ \sum_{i=1}^{3} a_{3i}a_{2i} \int_0^t \sigma^f(u,T) \int_u^T \sigma^\lambda(u,s)dsdu + \sum_{i=1}^{3} a_{3i}a_{2i} \int_0^t \sigma^\lambda(u,T) \int_u^T \sigma^f(u,s)dsdu \\ &+ \sum_{i=1}^{3} a_{3i} \int_0^t \sigma^f(u,T)d\tilde{W}_i(u) + \sum_{i=1}^{3} a_{2i} \int_0^t \sigma^\lambda(u,T)d\tilde{W}_i(u), \qquad (6.2)\end{aligned}$$

where for *notational convenience*, ω is omitted from $\sigma^f(t,T,\omega)$, $\sigma^\lambda(t,T,\omega)$, etc. Using equation (6.1a), this can be written as

$$\begin{aligned}\Delta f^d(t,T,\omega) = &\sum_{i=1}^{3} a_{3i}^2 \alpha_f(t,T) \int_0^t \sigma^f(u,t)\Big(\int_u^t \sigma^f(u,s)ds + \int_t^T \sigma^f(u,s)ds\Big)du \\ &+ \sum_{i=1}^{3} a_{2i}^2 \alpha_\lambda(t,T) \int_0^t \sigma^\lambda(u,t)\Big(\int_u^t \sigma^\lambda(u,s)ds + \int_t^T \sigma^\lambda(u,s)ds\Big)du \\ &+ \sum_{i=1}^{3} a_{3i}a_{2i}\alpha_f(t,T) \int_0^t \sigma^f(u,t)\Big(\int_u^t \sigma^\lambda(u,s)ds + \int_t^T \sigma^\lambda(u,s)ds\Big)du \\ &+ \sum_{i=1}^{3} a_{3i}a_{2i}\alpha_\lambda(t,T) \int_0^t \sigma^\lambda(u,t)\Big(\int_u^t \sigma^f(u,s)ds + \int_t^T \sigma^f(u,s)ds\Big)du \\ &+ \sum_{i=1}^{3} a_{3i}\alpha_f(t,T) \int_0^t \sigma^f(u,t)d\tilde{W}_i(u) + \sum_{i=1}^{3} a_{2i}\alpha_\lambda(t,T) \int_0^t \sigma^\lambda(u,t)d\tilde{W}_i(u), \\ &\qquad (6.3)\end{aligned}$$

which on further simplification yields

$$\begin{aligned}\Delta f^d(t,T,\omega) &= \sum_{i=1}^{3} a_{3i}^2 \alpha_f(t,T)\int_0^t \sigma^f(u,t)\int_u^t \sigma^f(u,s)dsdu + \sum_{i=1}^{3} a_{3i}\alpha_f(t,T)\int_0^t \sigma^f(u,t)d\tilde{W}_i(u)\\
&+ \sum_{i=1}^{3} a_{2i}^2 \alpha_\lambda(t,T)\int_0^t \sigma^\lambda(u,t)\int_u^t \sigma^\lambda(u,s)dsdu + \sum_{i=1}^{3} a_{3i}a_{2i}\alpha_\lambda(t,T)\int_0^t \sigma^f(u,t)\int_u^t \sigma^\lambda(u,s)dsdu\\
&+ \sum_{i=1}^{3} a_{3i}a_{2i}\alpha_\lambda(t,T)\int_0^t \sigma^\lambda(u,t)\int_u^t \sigma^f(u,s)dsdu + \sum_{i=1}^{3} a_{2i}\alpha_\lambda(t,T)\int_0^t \sigma^\lambda(u,t)d\tilde{W}_i(u)\\
&+ \sum_{i=1}^{3} a_{3i}^2 \alpha_f(t,T)\int_0^t \sigma^f(u,t)\int_t^T \sigma^f(u,s)dsdu + \sum_{i=1}^{3} a_{2i}^2 \alpha_\lambda(t,T)\int_0^t \sigma^\lambda(u,t)\int_t^T \sigma^\lambda(u,s)dsdu\\
&+ \sum_{i=1}^{3} a_{3i}a_{2i}\Big[\alpha_f(t,T)\int_0^t \sigma^f(u,t)\int_t^T \sigma^\lambda(u,s)dsdu + \alpha_\lambda(t,T)\int_0^t \sigma^\lambda(u,t)\int_t^T \sigma^f(u,s)dsdu\Big]\\
&+ \sum_{i=1}^{3} a_{3i}a_{2i}\Big[\alpha_f(t,T)\int_0^t \sigma^f(u,t)\int_u^t \sigma^\lambda(u,s)dsdu - \alpha_\lambda(t,T)\int_0^t \sigma^f(u,t)\int_u^t \sigma^\lambda(u,s)dsdu\Big].\end{aligned} \tag{6.4}$$

From the definition of the state variables in equations (4.12)-(4.15) , we can write equation (6.4) as

$$\begin{aligned}\Delta f^d(t,T,\omega) &= \alpha_f(t,T)[r(t,\omega) - f(0,t)] + \alpha_\lambda(t,T)[\lambda(t,\omega) - \lambda(0,t)] + [\alpha_f(t,T) - \alpha_\lambda(t,T)]S_3(t,\omega)\\
&+ \alpha_f(t,T)[\beta_f(t,T)\eta_1(t,\omega) + \beta_\lambda(t,T)\eta_3(t,\omega)]\\
&+ \alpha_\lambda(t,T)[\beta_\lambda(t,T)\eta_2(t,\omega) + \beta_f(t,T)\eta_3(t,\omega)],\end{aligned} \tag{6.5}$$

or equivalently setting $T = t+\tau$ in the above equation and defining $\Delta f^d(t,t+\tau,\omega) = \Delta f^d_\tau(t,\omega)$ yields

$$\begin{aligned}\Delta f^d_\tau(t,\omega) &= \alpha_f(t,t+\tau)\zeta_f(t,\omega) + \alpha_\lambda(t,t+\tau)\zeta_\lambda(t,\omega) + \beta_f(t,t+\tau)\alpha_f(t,t+\tau)\eta_1(t,\omega)\\
&+ \beta_\lambda(t,t+\tau)\alpha_\lambda(t,t+\tau)\eta_2(t,\omega) + [\alpha_f(t,t+\tau)\beta_\lambda(t,t+\tau) + \alpha_\lambda(t,t+\tau)\beta_f(t,t+\tau)]\eta_3(t,\omega)\\
&+ [\alpha_f(t,t+\tau) - \alpha_\lambda(t,t+\tau)]S_3(t,\omega),\end{aligned} \tag{6.6}$$

for the forward rate with fixed tenor τ.

It is observed that since $\alpha_f(t,t+\tau)$, $\alpha_\lambda(t,t+\tau)$, $\beta_f(t,t+\tau)$ and $\beta_\lambda(t,t+\tau)$ are deterministic functions, then for any τ, equation (6.6) gives the value of $\Delta f^d_\tau(t,\omega)$ and therefore the value of $f^d(t,t+\tau,\omega)$ as a linear combination of the state variables $\zeta_f(t,\omega)$, $\zeta_\lambda(t,\omega)$, $S_3(t,\omega)$ and $\eta_i(t,\omega)$ for $i = 1,2,3$.

In addition, equation (6.6) can be used to express the state variables as a linear combination of a finite set of forward rates. For example, for this we fix six tenors

$0 \leq \varsigma_1 \leq ... \leq \varsigma_6$ and setting $\tau = \varsigma_i$ gives rise to the system

$$\begin{bmatrix} \Delta f^d_{\varsigma_1}(t,\omega) \\ \Delta f^d_{\varsigma_2}(t,\omega) \\ \vdots \\ \Delta f^d_{\varsigma_6}(t,\omega) \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{16} \\ a_{21} & a_{22} & \dots & a_{26} \\ \vdots & \vdots & \ddots & \\ a_{61} & a_{62} & \dots & a_{66} \end{bmatrix}}_{\Lambda(t,\varsigma)} \begin{bmatrix} \zeta_f(t,\omega) \\ \zeta_\lambda(t,\omega) \\ \eta_1(t,\omega) \\ \eta_2(t,\omega) \\ \eta_3(t,\omega) \\ S_3(t,\omega) \end{bmatrix}, \tag{6.7}$$

where

$$\begin{aligned} &a_{i1} = \alpha_f(t,t+\varsigma_i), \quad a_{i2} = \alpha_\lambda(t,t+\varsigma_i), \quad a_{i3} = \beta_f(t,t+\varsigma_i)\alpha_f(t,t+\varsigma_i) \\ &a_{14} = \beta_\lambda(t,t+\varsigma_i)\alpha_\lambda(t,t+\varsigma_i), \quad a_{i5} = [\alpha_f(t,t+\varsigma_i)\beta_\lambda(t,t+\varsigma_i) + \alpha_\lambda(t,t+\varsigma_i)\beta_f(t,t+\varsigma_i)] \\ &a_{i6} = [\alpha_f(t,t+\varsigma_i) - \alpha_\lambda(t,t+\varsigma_i)] \quad \text{for} \quad i = 1,2,...,6. \end{aligned}$$

Assuming that the determinant of matrix $\Lambda(t,\varsigma)$ exists and is not equal to zero i.e, $det\Lambda(t,\varsigma) \neq 0$, this system of equations is invertible. We will follow the approach used in Chiarella and Kwon (2003) where they showed in their proposition 2 that if the matrix $\Lambda(t,\varsigma)$ is invertible, then the corresponding HJM model admits an affine Markovian realization in terms of the forward rates $f^d(t,t+\varsigma_i,\omega)$ for $i = 1,2,...6$. We can then write the state variables as linear combinations of forward rates $f^d_{\varsigma_1}(t,\omega)$, ..., $f^d_{\varsigma_6}(t,\omega)$ in the form,

$$\begin{bmatrix} \zeta_f(t,\omega) \\ \zeta_\lambda(t,\omega) \\ \eta_1(t,\omega) \\ \eta_2(t,\omega) \\ \eta_3(t,\omega) \\ S_3(t,\omega) \end{bmatrix} = \Lambda(t,\varsigma)^{-1} \begin{bmatrix} \Delta f^d_{\varsigma_1}(t,\omega) \\ \Delta f^d_{\varsigma_2}(t,\omega) \\ \vdots \\ \Delta f^d_{\varsigma_6}(t,\omega) \end{bmatrix}. \tag{6.8}$$

Equation (6.6) can therefore be used to write forward rates of all maturities in terms of a finite set and therefore the entire forward rate curve is parameterized by a set of fixed tenor forward rates. This offers the connection between the defaultable bond price and the market observed variables.

7 Generalized Defaultable HJM

7.1 Multifactor Model

We seek to generalize the results of the previous sections to allow for n Wiener processes driving the forward rate and the credit spread. Equations (3.9a) and (3.9b) can then be expressed as

$$df(t,T,\omega) = \alpha^f(t,T,\omega)dt + \sum_{i=1}^{n} \sigma_i^f(t,T,\omega)dW_i^f(t), \tag{7.1a}$$

$$d\lambda(t,T,\omega) = \alpha^\lambda(t,T,\omega)dt + \sum_{i=1}^{n} \sigma_i^\lambda(t,T,\omega)dW_i^\lambda(t), \tag{7.1b}$$

to allow for various factors in the economy or the market to drive the dynamics of the forward rate and credit spread. As in section 3.2, we assume that ω is the same under both volatility functions $\sigma_i^f(t,T,\omega)$ and $\sigma_i^\lambda(t,T,\omega)$ and is driven by the stochastic process V_i whose dynamics follow the SDE

$$dV_i(t) = \alpha_i^V(V,t)dt + \sigma_i^V(V,t)dW_i^V(t), \tag{7.2}$$

for $1 \leq i \leq n$.

For simplicity, it is further assumed that

$$\mathbb{E}\big[dW_i^x \cdot dW_j^y\big] = \begin{cases} \delta_{ij}\rho_i^{xy}dt & \text{if } x \neq y, \\ \delta_{ij}dt & \text{if } x = y, \end{cases} \tag{7.3}$$

where $x, y \in \{v, \lambda, f\}$, $1 \leq i, j \leq n$ and $\rho_i^{xy} \neq 0$ for all i. Defining $\rho_i^{12} = \rho_i^{v\lambda}$, $\rho_i^{13} = \rho_i^{vf}$ and $\rho_i^{23} = \rho_i^{\lambda f}$, the uncorrelated Wiener processes $W_i(t)$ can be obtained from the following relationships

$$W_i^v(t) = W_i(t), \tag{7.4a}$$

$$W_i^\lambda(t) = \rho_i^{12}W_i(t) + \sqrt{1-(\rho_i^{12})^2}W_{n+i}(t), \tag{7.4b}$$

$$W_i^f(t) = \rho_i^{13}W_i(t) + \frac{\rho_i^{23}-\rho_i^{12}\rho_i^{13}}{\sqrt{1-(\rho_i^{12})^2}}W_{n+i}(t) + \sqrt{\frac{1-(\rho_i^{12})^2-(\rho_i^{13})^2-(\rho_i^{23})^2+2\rho_i^{12}\rho_i^{13}\rho_i^{23}}{1-(\rho_i^{12})^2}}W_{2n+i}(t). \tag{7.4c}$$

The SDE's (7.1a), (7.1b) and (7.2) can then be expressed as

$$df(t,T,\omega) = \alpha^f(t,T,\omega)dt + \sum_{i=1}^{3n} \tilde{\sigma}_i^f(t,T,\omega)dW_i(t), \tag{7.5a}$$

$$d\lambda(t,T,\omega) = \alpha^\lambda(t,T,\omega)dt + \sum_{i=1}^{3n} \tilde{\sigma}_i^\lambda(t,T,\omega)dW_i(t), \tag{7.5b}$$

$$dV_i(t) = \alpha_i^V(V,t)dt + \tilde{\sigma}_i^V(t,T,\omega)dW_i(t), \quad 1 \le i \le n \tag{7.5c}$$

respectively and the volatility functions $\tilde{\sigma}_i^f(t,T,\omega)$, $\tilde{\sigma}_i^\lambda(t,T,\omega)$ and $\tilde{\sigma}_i^V(t,T,\omega)$ are given by

$$\begin{aligned}
&\tilde{\sigma}_i^f(t,T,\omega) = \rho_i^{13}\sigma_i^f(t,T,\omega), \quad \tilde{\sigma}_{n+i}^f(t,T,\omega) = \frac{\rho_i^{23} - \rho_i^{12}\rho_i^{13}}{\sqrt{1-(\rho_i^{12})^2}}\sigma_i^f(t,T,\omega),\\
&\tilde{\sigma}_{2n+i}^f(t,T,\omega) = \sqrt{\frac{1-(\rho_i^{12})^2-(\rho_i^{13})^2-(\rho_i^{23})^2+2\rho_i^{12}\rho_i^{13}\rho_i^{23}}{1-(\rho_i^{12})^2}}\sigma_i^f(t,T,\omega),\\
&\tilde{\sigma}_i^\lambda(t,T,\omega) = \rho_i^{12}\sigma_i^\lambda(t,T,\omega), \quad \tilde{\sigma}_{n+i}^\lambda(t,T,\omega) = \sqrt{1-(\rho_i^{12})^2}\sigma_i^\lambda(t,T,\omega),\\
&\tilde{\sigma}_i^V(t,T,\omega) = \sigma_i^V(V,t).
\end{aligned} \tag{7.6}$$

We observe that $\tilde{\sigma}_{2n+i}^\lambda(t,T,\omega) = 0$, $\tilde{\sigma}_{n+i}^V(t,T,\omega) = 0$ and $\tilde{\sigma}_{2n+i}^V(t,T,\omega) = 0$.

From definition 3.2, we observe using equation (3.5) that $f^d(t,T,\omega) = f(t,T,\omega) + \lambda(t,T,\omega)$. Then, given equations (7.5a), (7.5b) and the Girsanov's theorem as previously applied in subsection 3.4, we can derive the no-arbitrage drift restriction condition similar to 3.41

$$\begin{aligned}
\alpha^d(t,T,\omega) = &-\sum_{i=1}^{3n}\phi_i(t)\tilde{\sigma}_i^f(t,T,\omega) - \sum_{i=1}^{3n}\phi_i(t)\tilde{\sigma}_i^\lambda(t,T,\omega)\\
&+\sum_{i=1}^{3n}\tilde{\sigma}_i^f(t,T,\omega)\int_t^T\tilde{\sigma}_i^f(t,s,\omega)ds + \sum_{i=1}^{3n}\tilde{\sigma}_i^\lambda(t,T,\omega)\int_t^T\tilde{\sigma}_i^\lambda(t,s,\omega)ds\\
&+\sum_{i=1}^{3n}\tilde{\sigma}_i^\lambda(t,T,\omega)\int_t^T\tilde{\sigma}_i^f(t,s,\omega)ds + \sum_{i=1}^{3n}\tilde{\sigma}_i^f(t,T,\omega)\int_t^T\tilde{\sigma}_i^\lambda(t,s,\omega)ds.
\end{aligned} \tag{7.7}$$

Then, the time t instantaneous defaultable forward rate $f^d(t,T,\omega)$ in the *risk-neutral* n-factor HJM model is a stochastic process evolving according to the stochastic integral equation

$$f^d(t,T,\omega) = f^d(0,T) + \sum_{i=1}^{3n}\int_0^t\tilde{\sigma}_i^d(u,T,\omega)\int_u^T\tilde{\sigma}_i^d(u,s,\omega)dsdu + \sum_{i=1}^{3n}\int_0^t\tilde{\sigma}_i^d(u,T,\omega)d\tilde{W}_i(u), \tag{7.8}$$

where $0 \leq t \leq T$, $\tilde{W}_i = W_i - \int_0^t \phi_i(u)du$ and $\tilde{W}_i$ are independent standard $\tilde{\mathbb{P}}-$Wiener processes given that $0 \leq i \leq 3n$.

Similarly, using a generalisation of condition 3.43 to $3n$ we observe that the intensity rate dynamics evolve according to the stochastic integral equation

$$\lambda(t,\omega) = \lambda(0,t) + \sum_{i=1}^{3n} \int_0^t \tilde{\sigma}_i^{\lambda}(u,t,\omega) \int_u^t \tilde{\sigma}_i^{\lambda}(u,s,\omega)dsdu + \sum_{i=1}^{3n} \int_0^t \tilde{\sigma}_i^{\lambda}(u,t,\omega) \int_u^t \tilde{\sigma}_i^{f}(u,s,\omega)dsdu$$
$$+ \sum_{i=1}^{3n} \int_0^t \tilde{\sigma}_i^{f}(u,t,\omega) \int_u^t \tilde{\sigma}_i^{\lambda}(u,s,\omega)dsdu + \sum_{i=1}^{3n} \int_0^t \tilde{\sigma}_i^{\lambda}(u,t,\omega)d\tilde{W}_i(u). \tag{7.9}$$

. Under the risk-neutral measure, equation (7.5c) for stochastic volatility is driven by the stochastic differential equation

$$dV_i(t) = \big[\kappa_i(\bar{V}_i - V_i(t)) + \phi_i(t)\bar{\sigma}_i^V V_i^{\gamma}(t)\big]dt + \bar{\sigma}_i^V V_i^{\gamma}(t)d\tilde{W}_i(t), \tag{7.10}$$

for $1 \leq i \leq n$, $\gamma = 0.5$ and given that $\bar{\sigma}_i^V$ are constants.

7.2 Markovian System and Bond Pricing

The volatility functions in equation (7.6) satisfy the Markovian condition given in Inui and Kijima (1998) and Chiarella and Kwon (2001a). Consequently, the corresponding HJM model transforms to a finite dimensional Markovian system. The results obtained in section 4 with stochastic volatility remain valid for the multi-factor case in this section.

We observe that the bond price dynamics in the spirit of equation (A.4) satisfies

$$P^d(t,T) = \mathcal{R}(t)\text{exp}\Big[-\sum_{i=1}^{3n}\Big(\int_t^T f^d(0,s)ds + \int_t^T \int_0^t \tilde{\sigma}_i^{d*}(u,s,\omega)duds$$
$$+ \int_t^T \int_0^t \tilde{\sigma}_i^{d}(u,s,\omega)d\tilde{W}_i(u)ds\Big)\Big], \tag{7.11}$$

where the volatility functions $\tilde{\sigma}_i^{d*}(t,T,\omega) = \tilde{\sigma}_i^{d}(t,T,\omega)\int_t^T \tilde{\sigma}_i^{d}(t,s,\omega)ds$ satisfy equation (7.6). Following the idea in Proposition 4.1, we define some new state variables

$$d\eta_1^{'}(t,\omega) = \Big(\sum_{i=1}^{3n} a_{3i}^2\bar{\sigma}_f^2 r(t,\omega)V(t) - 2\kappa_f\eta_1^{'}(t,\omega)\Big)dt, \tag{7.12a}$$

$$d\eta_2^{'}(t,\omega) = \Big(\sum_{i=1}^{3n} a_{2i}^2\bar{\sigma}_\lambda^2 \lambda(t,\omega)V(t) - 2\kappa_\lambda\eta_2^{'}(t,\omega)\Big)dt, \tag{7.12b}$$

$$d\eta_3^{'}(t,\omega) = \Big(\sum_{i=1}^{3n} a_{2i}a_{3i}\bar{\sigma}_f\bar{\sigma}_\lambda\sqrt{r(t,\omega)\lambda(t,\omega)}V(t) - (\kappa_f+\kappa_\lambda)\eta_3^{'}(t,\omega)\Big)dt. \tag{7.12c}$$

In addition, we define

$$S_3^{'}(t,\omega) = \sum_{i=1}^{3n} a_{2i}a_{3i} \int_0^t \sigma^f(u,t,\omega) \int_u^t \sigma^\lambda(u,v,\omega)dvdu, \tag{7.13}$$

such that

$$dS_3^{'}(t,\omega) = \Big(\eta_3^{'}(t,\omega) - \kappa_f S_3^{'}(t,\omega)\Big)dt.$$

Then, we have the following generalized version of Theorem 4.2.

Theorem 7.1 *Given that under the multifactor model the bond price dynamics satisfy equation* (7.11)*, then the bond price is an exponential affine function of the form*

$$P^d(t,T) = \frac{\bar{P}^d(0,T)}{\bar{P}^d(0,t)} \exp\Big(- D^{'}(t,T,\omega) - \beta_f(t,T)\zeta_f(t,\omega) - \beta_\lambda(t,T)\zeta_\lambda(t,\omega)\Big), \tag{7.14}$$

where the coefficients are defined by

$$\zeta_f(t,\omega) = r(t,\omega) - f(0,t), \quad \zeta_\lambda(t,\omega) = \lambda(t,\omega) - \lambda(0,t),$$

$$\begin{aligned} D^{'}(t,T,\omega) = & -ln\mathcal{R}(t)\frac{1}{2}\beta_f^2(t,T)\eta_1^{'}(t,\omega) + \frac{1}{2}\beta_\lambda^2(t,T)\eta_2^{'}(t,\omega) + A(t,T)\eta_3^{'}(t,\omega) \\ & + \big[\beta_f(t,T) - \beta_\lambda(t,T)\big]S_3^{'}(t,\omega), \end{aligned}$$

given that

$$A(t,T) = \frac{1}{\kappa_f}\beta_f(t,T) + \frac{1}{\kappa_\lambda}\beta_\lambda(t,T) + \Big(\frac{1}{\kappa_f} + \frac{1}{\kappa_\lambda}\Big)\Big(\frac{1}{\kappa_f + \kappa_\lambda}\Big)\Big(1 - e^{-(\kappa_f+\kappa_\lambda)(T-t)}\Big).$$

Proof: See Appendix E. ■

We observe that the results of this theorem are similar to equation (4.21) in Theorem 4.2, the main difference being that $1 \leq i \leq 3n$ and in the definition of the volatility functions. In addition, the state variables defined in section 6 can also be expressed as functions of forward rates in the multi-factor framework, albeit with straight forward modifications to allow for the n-factors.

8 Conclusion

In this paper we have shown how the stochastic dynamics of the defaultable Heath-Jarrow-Morton interest rate framework with stochastic volatility can be reduced to a Markovian form. This allows us to express defaultable bond price in terms of the underlying state variables, thereby reducing the computational time required for the calculation of interest rate derivative prices as Monte carlo simulation would be required only for calculation of option prices.

The paper also established an explicit formula for the defaultable bond price in the presence of stochastic volatility, using the approach developed in the default-free framework by Ritchken and Sankarasubramanian (1995), Inui and Kijima (1998) and Chiarella and Kwon (2001a) by incorporating default risk. In particular, it was shown that the defaultable bond price takes an exponential affine form in the sense of the square root affine volatility models considered in Duffie and Kan (1996). The only non-market traded quantities in the stochastic dynamics are the stochastic volatility quantities.

Some numerical results have been given indicating how the level of the volatility of volatility, default intensity and correlation between the noises driving the defaultable forward rates, credit spreads and the stochastic volatility affect the defaultable bond price and bond return values. The paper also attempts to provide a link between the state variables and the market observed quantities of the forward rates. This would be of significant value in implementing the model in practice and further research into the practical implementation, calibration and evaluation of these models remain an on-going project.

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A Proof of Proposition 3.1

From the definition of the defaultable bond in equation (3.7) we recall that

$$P^d(t,T) = \mathcal{R}(t)\bar{P}^d(t,T,\omega), \tag{A.1}$$

where $\bar{P}^d(t,T,\omega)$, the pseudo-bond is as defined in equation (3.6) and $\mathcal{R}(t)$ is the remainder of all fractional default losses at time t. We observe that the pre-default value of the bond is given by

$$P^d(t-,T,\omega) = \mathcal{R}(t-)\bar{P}^d(t,T,\omega), \tag{A.2}$$

where $\bar{P}^d(t,T,\omega)$ is a continuous function as per the definition of the pseudo-bond.

Applying Itô’s lemma to equation (A.2) yields

$$dP^d(t,T) = \bar{P}^d(t,T,\omega)d\mathcal{R}(t) + \mathcal{R}(t-)d\bar{P}^d(t,T,\omega) + d\mathcal{R}(t)d\bar{P}^d(t,T,\omega),$$

which is equivalent to

$$\begin{aligned} dP^d(t,T) &= \mathcal{R}(t-)d\bar{P}^d(t,T,\omega) + d\mathcal{R}(t)\big(\bar{P}^d(t,T,\omega) + d\bar{P}^d(t,T,\omega)\big), \\ &= \mathcal{R}(t-)d\bar{P}^d(t,T,\omega) + \bar{P}^d(t,T,\omega)d\mathcal{R}(t), \end{aligned}$$

since by definition of continuity $\bar{P}^d(t,T,\omega) = \bar{P}^d(t,T,\omega) + d\bar{P}^d(t,T,\omega)$. Using equation (A.2), we observe that

$$dP^d(t,T) = \mathcal{R}(t-)\bar{P}^d(t-,T,\omega)\frac{d\bar{P}^d(t,T,\omega)}{\bar{P}^d(t-,T,\omega)} + \bar{P}^d(t,T,\omega)\mathcal{R}(t-)\frac{d\mathcal{R}(t)}{\mathcal{R}(t-)},$$

and given that $\bar{P}^d(t-,T,\omega) = \bar{P}^d(t,T,\omega)$ this reduces to

$$dP^d(t,T) = P^d(t-,T,\omega)\underbrace{\frac{d\bar{P}^d(t,T,\omega)}{\bar{P}^d(t,T,\omega)}}_{A} + P^d(t-,T,\omega)\underbrace{\frac{d\mathcal{R}(t)}{\mathcal{R}(t-)}}_{B}. \tag{A.3}$$

It remains to obtain the stochastic differential equation for the pseudo-bond $\bar{P}^d(t,T,\omega)$.

On substituting (3.23) into (3.4) we have

$$\bar{P}^d(t,T,\omega) = \exp\bigg(- \int_t^T f^d(0,s)ds - \int_t^T \int_0^t \alpha^d(u,s,\omega)duds - \sum_{i=1}^3 \int_t^T \int_0^t \tilde{\sigma}_i^d(u,s,\omega)dW_i(u)ds \bigg). \tag{A.4}$$

In addition, following equation (3.23) we observe that

$$\int_t^T f^d(t,s,\omega)ds = \int_t^T f^d(0,s)ds + \int_t^T \int_0^t \alpha^d(u,s,\omega)duds + \sum_{i=1}^3 \int_t^T \int_0^t \tilde{\sigma}_i^d(u,s,\omega)dW_i(u)ds, \tag{A.5}$$

which on using stochastic Fubini's theorem yields

$$\int_t^T f^d(t,s,\omega)ds = \int_t^T f^d(0,s)ds + \int_0^t \int_t^T \alpha^d(u,s,\omega)dsdu + \sum_{i=1}^3 \int_0^t \int_t^T \tilde{\sigma}_i^d(u,s,\omega)dsdW_i(u). \tag{A.6}$$

This can be written as

$$\begin{aligned}\int_t^T f^d(t,s,\omega)ds = {} & \int_0^T f^d(0,s)ds + \int_0^t \int_u^T \alpha^d(u,s,\omega)dsdu + \sum_{i=1}^3 \int_0^t \int_u^T \tilde{\sigma}_i^d(u,s,\omega)dsdW_i(u) \\ & - \int_0^t f^d(0,s)ds - \int_0^t \int_u^t \alpha^d(u,s,\omega)dsdu - \sum_{i=1}^3 \int_0^t \int_u^t \tilde{\sigma}_i^d(u,s,\omega)dsdW_i(u),\end{aligned}$$

or equivalently,

$$\begin{aligned}\int_t^T f^d(t,s,\omega)ds = {} & -\ln P^d(0,T) - \int_0^t r^d(s,\omega)ds + \int_0^t \int_u^T \alpha^d(u,s,\omega)dsdu \\ & + \sum_{i=1}^3 \int_0^t \int_u^T \tilde{\sigma}_i^d(u,s,\omega)dsdW_i(u),\end{aligned} \tag{A.7}$$

where we have used the definition of the defaultable bond as a function of the pseudo-bond to show that

$$P^d(0,T) = \mathcal{R}(0)\bar{P}^d(0,T) = \bar{P}^d(0,T).$$

We define the log bond price $\bar{B}^d(t,T,\omega) = \ln\bar{P}^d(t,T,\omega)$. Then, using equations (A.7) and (3.4) the log bond price can then be expressed as the stochastic integral equation

$$\begin{aligned}\bar{B}^d(t,T,\omega) = \int_0^t r^d(s,\omega)ds + \ln P^d(0,T) - \int_0^t \int_u^T \alpha^d(u,s,\omega)dsdu \\ - \sum_{i=1}^{3} \int_0^t \int_u^T \tilde{\sigma}_i^d(u,s,\omega)dsdW_i(u).\end{aligned} \tag{A.8}$$

Equivalently, this can be expressed as the stochastic differential equation

$$d\bar{B}^d(t,T,\omega) = [r^d(t,\omega) - \alpha_B^d(t,T,\omega)]dt + \sum_{i=1}^{3} \tilde{\sigma}_{B,i}^d(t,T,\omega)dW_i(t), \tag{A.9}$$

where

$$\alpha_B^d(u,T,\omega) = \int_u^T \alpha^d(u,s,\omega)ds, \quad \text{and} \quad \tilde{\sigma}_{B,i}^d(u,T,\omega) = -\int_u^T \tilde{\sigma}_i^d(u,s,\omega)ds. \tag{A.10a}$$

It can then be seen that the pseudo-bond dynamics satisfy the stochastic differential equation

$$d\bar{P}^d(t,T,\omega) = \bar{P}^d(t,T,\omega)\big(r^d(t,\omega) + b^d(t,T,\omega)\big)dt + \bar{P}^d(t,T,\omega)\sum_{i=1}^{3} \tilde{\sigma}_{B,i}^d(t,T,\omega)dW_i(t), \tag{A.11}$$

where the coefficients in the drift and diffusion are given by

$$b^d(t,T,\omega) = -\alpha_B^d(t,T,\omega) + \frac{1}{2}\sum_{i=1}^{3} \big(\tilde{\sigma}_{B,i}^d(t,T,\omega)\big)^2, \tag{A.12}$$

$$\tilde{\sigma}_{B,i}^d(t,T,\omega) = -\int_t^T \tilde{\sigma}_i^d(t,s,\omega)ds. \tag{A.13}$$

Substituting equation(A.11) into (A.3) then yields the stochastic differential equation for the defaultable bond

$$\frac{dP^d(t,T)}{P^d(t-,T,\omega)} = \big(r^d(t,\omega) + b^d(t,T,\omega)\big)dt + \sum_{i=1}^{3} \tilde{\sigma}_{B,i}^d(t,T,\omega)dW_i(t) - d\bar{N}(t). \tag{A.14}$$

Equivalently, we note that $d\bar{M}(t) = d\bar{N}(t) - q(t)h(t)dt$ such that $\bar{M}(t)$ is a martingale. Then the defaultable price dynamics can alternatively be written as

$$\frac{dP^d(t,T)}{P^d(t-,T,\omega)} = \big(r^d(t,\omega) + b^d(t,T,\omega) - q(t)h(t)\big)dt + \sum_{i=1}^{3} \tilde{\sigma}^d_{B,i}(t,T,\omega)dW_i(t) - d\bar{M}(t). \tag{A.15}$$

Hence the proof of Proposition 3.1. ♦

B Hedging Argument in the Defaultable Bond Market.

From the version of the Girsanov's Theorem presented in Björk, Kabanov and Runggaldier (1997) (Theorem 3.12) which they applied in the default-free framework, the defaultable bond price dynamics in equation (3.31) under the risk-neutral measure can be written as

$$\begin{aligned}\frac{dP^d(t,T)}{P^d(t-,T,\omega)} &= \big[r^d(t,\omega) + b^d(t,T,\omega)\big]dt + \sum_{i=1}^{3} \tilde{\sigma}^d_{B,i}(t,T,\omega)[d\tilde{W}_i(t) + \phi_i(t)dt] \\ &\quad - q(t)[dN(t) - \tilde{h}(t)dt] - \tilde{h}(t)q(t)dt. \end{aligned}\tag{B.1}$$

This can be written as

$$\begin{aligned}\frac{dP^d(t,T)}{P^d(t-,T,\omega)} &= \big[r^d(t,\omega) + b^d(t,T,\omega) + \sum_{i=1}^{3} \phi_i(t)\tilde{\sigma}^d_{B,i}(t,T,\omega) - \tilde{h}(t)q(t)\big]dt \\ &\quad + \sum_{i=1}^{3} \tilde{\sigma}^d_{B,i}(t,T,\omega)d\tilde{W}_i(t) - q(t)d\tilde{M}(t), \end{aligned}\tag{B.2}$$

where $d\tilde{M}(t) = dN(t) - \tilde{h}(t)dt$ is a martingale under the risk-neutral measure.

Using the approach in Björk et al. (1997), from the fundamental theorem of asset pricing we know that a measure $\tilde{\mathbb{P}}$ is a risk-neutral measure if and only if the discounted bond price is a martingale such that the bond dynamics are of the form

$$dP^d(t,T) = P^d(t,T)r(t,\omega) + d\tilde{V}(t), \tag{B.3}$$

where $\tilde{V}$ is a $\tilde{\mathbb{P}}-$local martingale. Then, comparing the drifts of equations (B.2) and (B.3) we observe that

$$r^d(t,\omega) + b^d(t,T,\omega) + \sum_{i=1}^{3} \phi_i(t)\tilde{\sigma}^d_{B,i}(t,T,\omega) - \tilde{h}(t)q(t) = r(t,\omega).$$

Hence the proof. ■

C Proof of Proposition 4.1

From equation (3.47) and assumption 4.1, it follows that $r^d(t,\omega)$ follows the stochastic differential equation

$$dr^d(t,\omega) = \big[f_2(0,t) + \frac{\partial}{\partial t}\sum_{j=1}^{4} S_j(t,\omega) - \kappa_f \psi_1(t,\omega) - \kappa_\lambda \psi_2(t,\omega)\big]dt$$
$$+ \Big(\sum_{i=1}^{3} a_{3i}\bar{\sigma}_f \sqrt{r(t,\omega)V(t)} + \sum_{i=1}^{3} a_{2i}\bar{\sigma}_\lambda \sqrt{\lambda(t,\omega)V(t)}\Big) d\tilde{W}_i(t). \tag{C.1}$$

From the definition of the short rate and intensity processes in equations (3.50) and (**??**) respectively, the state variables $\psi_1(t,\omega)$ and $\psi_2(t,\omega)$ can be expressed as

$$\psi_1(t,\omega) = r(t,\omega) - f(0,t) - S_1(t,\omega), \tag{C.2}$$

$$\psi_2(t,\omega) = \lambda(t,\omega) - \lambda(0,t) - \sum_{j=2}^{4} S_j(t,\omega). \tag{C.3}$$

In addition, using assumption 4.1 for the volatility functions, we can write the differentials of the state variables $S_j(t,\omega)$, $j = 1,2,3,4$ as

$$dS_1(t,\omega) = \Big(\eta_1(t,\omega) - \kappa_f S_1(t,\omega)\Big)dt, \tag{C.4a}$$

$$dS_2(t,\omega) = \Big(\eta_2(t,\omega) - \kappa_\lambda S_2(t,\omega)\Big)dt, \tag{C.4b}$$

$$dS_3(t,\omega) = \Big(\eta_3(t,\omega) - \kappa_f S_3(t,\omega)\Big)dt, \tag{C.4c}$$

$$dS_4(t,\omega) = \Big(\eta_3(t,\omega) - \kappa_\lambda S_4(t,\omega)\Big)dt, \tag{C.4d}$$

where we have defined the additional subsidiary state variables $\eta_j(t,\omega)$ as

$$\eta_1(t,\omega) = \sum_{i=1}^{3} a_{3i}^2 \int_0^t \bar{\sigma}_f^2 r(u,\omega)V(u)e^{-2\kappa_f(t-u)}du, \tag{C.5a}$$

$$\eta_2(t,\omega) = \sum_{i=1}^{3} a_{2i}^2 \int_0^t \bar{\sigma}_\lambda^2 \lambda(u,\omega)V(u)e^{-2\kappa_\lambda(t-u)}du, \tag{C.5b}$$

$$\eta_3(t,\omega) = \sum_{i=1}^{3} a_{2i}a_{3i} \int_0^t \bar{\sigma}_f\bar{\sigma}_\lambda \sqrt{r(u,\omega)\lambda(u,\omega)}V(u)e^{-(\kappa_f+\kappa_\lambda)(t-u)}du, \tag{C.5c}$$

which follows the differential equations

$$d\eta_1(t,\omega) = \Big(\sum_{i=1}^{3} a_{3i}^2 \bar{\sigma}_f^2 r(t,\omega)V(t) - 2\kappa_f \eta_1(t,\omega)\Big)dt, \tag{C.6a}$$

$$d\eta_2(t,\omega) = \Big(\sum_{i=1}^{3} a_{2i}^2 \bar{\sigma}_\lambda^2 \lambda(t,\omega)V(t) - 2\kappa_\lambda \eta_2(t,\omega)\Big)dt, \tag{C.6b}$$

$$d\eta_3(t,\omega) = \Big(\sum_{i=1}^{3} a_{2i}a_{3i} \bar{\sigma}_f \bar{\sigma}_\lambda \sqrt{r(t,\omega)\lambda(t,\omega)}V(t) - (\kappa_f + \kappa_\lambda)\eta_3(t,\omega)\Big)dt, \tag{C.6c}$$

Substituting equations (C.2) and (C.3) and their differentials into equation (C.1) yields

$$\begin{aligned} dr^d(t,\omega) = \big[&f_2(0,t) + \eta_1(t,\omega) - \kappa_f S_1(t,\omega) + \eta_2(t,\omega) - \kappa_\lambda S_2(t,\omega) \\ &+ 2\eta_3(t,\omega) - \kappa_f S_3(t,\omega) - \kappa_\lambda S_4(t,\omega) - \kappa_f \psi_1(t,\omega) - \kappa_\lambda \psi_2(t,\omega)\big]dt \\ &+ \Big(\sum_{i=1}^{3} a_{3i}\bar{\sigma}_f \sqrt{r(t,\omega)V(t)} + \sum_{i=1}^{3} a_{2i}\bar{\sigma}_\lambda \sqrt{\lambda(t,\omega)V(t)}\Big)d\tilde{W}_i(t). \end{aligned} \tag{C.7}$$

From equations (2.12) and (3.50), we observe that

$$r(t,\omega) - f(0,t) = S_1(t,\omega) + \psi_1(t,\omega), \quad \text{and} \quad \lambda(t,\omega) - \lambda(0,t) = \sum_{j=2}^{4} S_j(t,\omega) + \psi_2(t,\omega),$$

which on substituting into equation (C.7) we have

$$\begin{aligned} dr^d(t,\omega) = \big[&f_2(0,t) + \kappa_f f(0,t) + \kappa_\lambda \lambda(0,t) + \eta_1(t,\omega) + \eta_2(t,\omega) \\ &+ 2\eta_3(t,\omega) - (\kappa_f - \kappa_\lambda)S_3(t,\omega) - \kappa_f r(t,\omega) - \kappa_\lambda \lambda(t,\omega)\big]dt \\ &+ \Big(\sum_{i=1}^{3} a_{3i}\bar{\sigma}_f \sqrt{r(t,\omega)V(t)} + \sum_{i=1}^{3} a_{2i}\bar{\sigma}_\lambda \sqrt{\lambda(t,\omega)V(t)}\Big)d\tilde{W}_i(t). \end{aligned} \tag{C.8}$$

This can be rearranged to yield

$$\begin{aligned} dr^d(t,\omega) = \big[&f_2(0,t) + \kappa_f f(0,t) + \kappa_\lambda \lambda(0,t) + \eta_1(t,\omega) + \eta_2(t,\omega) \\ &+ 2\eta_3(t,\omega) - (\kappa_f - \kappa_\lambda)S_3(t,\omega) - (\kappa_f - \kappa_\lambda)\lambda(t,\omega) - \kappa_f r^d(t,\omega)\big]dt \\ &+ \Big(\sum_{i=1}^{3} a_{3i}\bar{\sigma}_f \sqrt{r(t,\omega)V(t)} + \sum_{i=1}^{3} a_{2i}\bar{\sigma}_\lambda \sqrt{\lambda(t,\omega)V(t)}\Big)d\tilde{W}_i(t). \end{aligned} \tag{C.9}$$

Defining a new coefficient $\theta_d(t,\omega)$ such that

$$\theta_d(t,\omega) = f_2(0,t) + \kappa_f f(0,t) + \kappa_\lambda \lambda(0,t),$$

gives the result in proposition 4.1. Hence the proof. ◆

D Defaultable Bond Price Formula

We recall from equation (3.4) that the price of the defaultable bond is given by

$$P^d(t,T) = \mathcal{R}(t)\bar{P}^d(t,T,\omega), \tag{D.1}$$

where $\bar{P}^d(t,T,\omega)$ defines the Pseudo-bond and the recovery process is

$$\mathcal{R}(t) := \prod_{\tau_i \leq t}(1 - q(\tau_i)).$$

In addition, the 'pseudo' bond is defined by

$$\bar{P}^d(t,T,\omega) = \exp\Big(- \int_t^T f^d(t,s,\omega)ds \Big). \tag{D.2}$$

We can rewrite the defaultable forward rate stochastic integral equation (3.23) as

$$f^d(t,T,\omega) = f^d(0,T) + \sum_{i=1}^{3}\Big[\int_0^t \tilde{\sigma}_i^{d*}(u,T,\omega)du + \int_0^t \tilde{\sigma}_i^{d}(u,T,\omega)d\tilde{W}_i(u)\Big], \tag{D.3}$$

where

$$\tilde{\sigma}_i^{d*}(t,T,\omega) = \tilde{\sigma}_i^{d}(t,T,\omega)\int_t^T \tilde{\sigma}_i^{d}(t,s,\omega)ds. \tag{D.4}$$

Then, equation (D.2) becomes

$$\begin{aligned}\bar{P}^d(t,T,\omega) = \exp\Big[- \sum_{i=1}^{3}\Big(\int_t^T f^d(0,s)ds + \int_t^T\int_0^t \tilde{\sigma}_i^{d*}(u,s,\omega)duds \\ + \int_t^T\int_0^t \tilde{\sigma}_i^{d}(u,s,\omega)d\tilde{W}_i(u)ds\Big)\Big].\end{aligned} \tag{D.5}$$

We define a new variable I such that

$$I = \int_t^T\int_0^t \tilde{\sigma}_i^{d*}(u,s,\omega)duds + \int_t^T\int_0^t \tilde{\sigma}_i^{d}(u,s,\omega)d\tilde{W}_i(u)ds. \tag{D.6}$$

By applying Fubini's theorem, this can be rewritten as

$$I = \underbrace{\int_0^t\int_t^T \tilde{\sigma}_i^{d*}(u,s,\omega)dsdu}_{I_1} + \underbrace{\int_0^t\int_t^T \tilde{\sigma}_i^{d}(u,s,\omega)dsd\tilde{W}_i(u)}_{I_2}, \tag{D.7}$$

i.e, $I = I_1 + I_2$. We note that,

$$\begin{aligned}\int_t^T \tilde{\sigma}_i^{d*}(u,s,\omega)ds &= \int_t^T \tilde{\sigma}_i^{d}(u,s,\omega)\int_u^t \tilde{\sigma}_i^{d}(u,v,\omega)dvds + \int_t^T \tilde{\sigma}_i^{d}(u,s,\omega)\int_t^s \tilde{\sigma}_i^{d}(u,v,\omega)dvds, \\ &= \int_t^T (\tilde{\sigma}_i^{f}(u,s,\omega) + \tilde{\sigma}_i^{\lambda}(u,s,\omega))\int_u^t (\tilde{\sigma}_i^{f}(u,v,\omega) + \tilde{\sigma}_i^{\lambda}(u,v,\omega))dvds \\ &+ \int_t^T (\tilde{\sigma}_i^{f}(u,s,\omega) + \tilde{\sigma}_i^{\lambda}(u,s,\omega))\int_t^s (\tilde{\sigma}_i^{f}(u,v,\omega) + \tilde{\sigma}_i^{\lambda}(u,v,\omega))dvds\end{aligned} \tag{D.8}$$

This can be written as,

$$
\begin{aligned}
\int_t^T \tilde{\sigma}_i^{d*}(u,s,\omega)ds &= \int_t^T \tilde{\sigma}_i^f(u,s,\omega)\int_u^t \tilde{\sigma}_i^f(u,v,\omega)dvds + \int_t^T \tilde{\sigma}_i^f(u,s,\omega)\int_t^s \tilde{\sigma}_i^f(u,v,\omega)dvds \\
&+ \int_t^T \tilde{\sigma}_i^\lambda(u,s,\omega)\int_u^t \tilde{\sigma}_i^\lambda(u,v,\omega)dvds + \int_t^T \tilde{\sigma}_i^\lambda(u,s,\omega)\int_t^s \tilde{\sigma}_i^\lambda(u,v,\omega)dvds \\
&+ \int_t^T \tilde{\sigma}_i^f(u,s,\omega)\int_u^t \tilde{\sigma}_i^\lambda(u,v,\omega)dvds + \int_t^T \tilde{\sigma}_i^f(u,s,\omega)\int_t^s \tilde{\sigma}_i^\lambda(u,v,\omega)dvds \\
&+ \int_t^T \tilde{\sigma}_i^\lambda(u,s,\omega)\int_u^t \tilde{\sigma}_i^f(u,v,\omega)dvds + \int_t^T \tilde{\sigma}_i^\lambda(u,s,\omega)\int_t^s \tilde{\sigma}_i^f(u,v,\omega)dvds.
\end{aligned}
\tag{D.9}
$$

In addition, we observe that

$$
\begin{aligned}
\int_t^T \tilde{\sigma}_i^f(u,s,\omega)\int_u^t \tilde{\sigma}_i^f(u,v,\omega)dvds &= B\int_t^T e^{-\kappa_f(s-u)}\int_u^t e^{-\kappa_f(v-u)}dvds, \\
\int_t^T \tilde{\sigma}_i^f(u,s,\omega)\int_t^s \tilde{\sigma}_i^f(u,v,\omega)dvds &= B\int_t^T e^{-\kappa_f(s-u)}\int_t^s e^{-\kappa_f(v-u)}dvds, \\
\int_t^T \tilde{\sigma}_i^\lambda(u,s,\omega)\int_u^t \tilde{\sigma}_i^\lambda(u,v,\omega)dvds &= C\int_t^T e^{-\kappa_\lambda(s-u)}\int_u^t e^{-\kappa_\lambda(v-u)}dvds, \\
\int_t^T \tilde{\sigma}_i^\lambda(u,s,\omega)\int_t^s \tilde{\sigma}_i^\lambda(u,v,\omega)dvds &= C\int_t^T e^{-\kappa_\lambda(s-u)}\int_t^s e^{-\kappa_\lambda(v-u)}dvds, \\
\int_t^T \tilde{\sigma}_i^f(u,s,\omega)\int_u^t \tilde{\sigma}_i^\lambda(u,v,\omega)dvds &= D\int_t^T e^{-\kappa_f(s-u)}\int_u^t e^{-\kappa_\lambda(v-u)}dvds, \\
\int_t^T \tilde{\sigma}_i^\lambda(u,s,\omega)\int_u^t \tilde{\sigma}_i^f(u,v,\omega)dvds &= D\int_t^T e^{-\kappa_\lambda(s-u)}\int_u^t e^{-\kappa_f(v-u)}dvds,
\end{aligned}
$$

where

$$B = a_{3i}^2 r(u,\omega)V(u), \quad C = a_{2i}^2\lambda(u,\omega)V(u) \quad \text{and} \quad D = a_{3i}a_{2i}V(u)\sqrt{\lambda(u,\omega)r(u,\omega)}.$$

Following the argument in Chiarella and Kwon (1999), it then follows that we can write equation (D.9) as

$$
\begin{aligned}
\int_t^T \tilde{\sigma}_i^{d*}(u,s,\omega)ds &= B\Big[\beta_f(t,T)e^{-\kappa_f(t-u)}\int_u^t e^{-\kappa_f(v-u)}dv + e^{-2\kappa_f(t-u)}\int_t^T e^{-\kappa_f(s-t)}\int_t^s e^{-\kappa_f(v-t)}dvds\Big] \\
&+ C\Big[\beta_\lambda(t,T)e^{-\kappa_\lambda(t-u)}\int_u^t e^{-\kappa_\lambda(v-u)}dv + e^{-2\kappa_\lambda(t-u)}\int_t^T e^{-\kappa_\lambda(s-t)}\int_t^s e^{-\kappa_\lambda(v-t)}dvds\Big] \\
&+ D\Big[\beta_f(t,T)e^{-\kappa_f(t-u)}\int_u^t e^{-\kappa_\lambda(v-u)}dv + \beta_\lambda(t,T)e^{-\kappa_\lambda(t-u)}\int_u^t e^{-\kappa_f(v-u)}dv \\
&+ e^{-(\kappa_f+\kappa_\lambda)(t-u)}\Big(\int_t^T e^{-\kappa_f(s-t)}\int_t^s e^{-\kappa_\lambda(v-t)}dvds + \int_t^T e^{-\kappa_\lambda(s-t)}\int_t^s e^{-\kappa_f(v-t)}dvds\Big)\Big],
\end{aligned}
\tag{D.10}
$$

where we define the deterministic functions in equation (4.20).

It can then be shown that

$$\int_t^T e^{-\kappa_f(s-t)} \int_t^s e^{-\kappa_\lambda(v-t)} dv ds = \frac{1}{\kappa_\lambda}\beta_f(t,T) + \frac{1}{\kappa_\lambda(\kappa_f+\kappa_\lambda)}\Big(1 - e^{-(\kappa_f+\kappa_\lambda)(T-t)}\Big),$$

and

$$\int_t^T e^{-\kappa_\lambda(s-t)} \int_t^s e^{-\kappa_f(v-t)} dv ds = \frac{1}{\kappa_f}\beta_\lambda(t,T) + \frac{1}{\kappa_f(\kappa_f+\kappa_\lambda)}\Big(1 - e^{-(\kappa_\lambda+\kappa_f)(T-t)}\Big).$$

We can then re-write equation (D.10) as

$$\begin{aligned}
\int_t^T \tilde{\sigma}_i^{d*}(u,s,\omega)ds &= \beta_f(t,T)\tilde{\sigma}_i^f(u,t,\omega)\int_u^t \tilde{\sigma}_i^f(u,v,\omega)dv + \beta_\lambda(t,T)\tilde{\sigma}_i^\lambda(u,t,\omega)\int_u^t \tilde{\sigma}_i^\lambda(u,v,\omega)dv \\
&\quad + \frac{1}{2}\beta_f^2(t,T)\tilde{\sigma}_i^{2f}(u,t,\omega) + \frac{1}{2}\beta_\lambda^2(t,T)\tilde{\sigma}_i^{2\lambda}(u,t,\omega) + \beta_f(t,T)\tilde{\sigma}_i^\lambda(u,t,\omega)\int_u^t \tilde{\sigma}_i^f(u,v,\omega)dv \\
&\quad + \beta_\lambda(t,T)\tilde{\sigma}_i^f(u,t,\omega)\int_u^t \tilde{\sigma}_i^\lambda(u,v,\omega)dv + \Big[\frac{1}{\kappa_\lambda}\beta_f(t,T) + \frac{1}{\kappa_f}\beta_\lambda(t,T) \\
&\quad + \Big(\frac{1}{\kappa_f} + \frac{1}{\kappa_\lambda}\Big)\Big(\frac{1}{\kappa_f+\kappa_\lambda}\Big)\Big(1 - e^{-(\kappa_f+\kappa_\lambda)(T-t)}\Big)\Big]\tilde{\sigma}_i^f(u,t,\omega)\tilde{\sigma}_i^\lambda(u,t,\omega) \qquad \text{(D.11)}
\end{aligned}$$

In addition, we observe that

$$\begin{aligned}
\int_t^T \tilde{\sigma}_i^d(u,s,\omega)ds &= \int_t^T \tilde{\sigma}_i^f(u,s,\omega)ds + \int_t^T \tilde{\sigma}_i^\lambda(u,s,\omega)ds \\
&= \beta_f(t,T)\tilde{\sigma}_i^f(u,t,\omega) + \beta_\lambda(t,T)\tilde{\sigma}_i^\lambda(u,t,\omega). \qquad \text{(D.12)}
\end{aligned}$$

We recall from equation (D.7) that,

$$I = \underbrace{\int_0^t \int_t^T \tilde{\sigma}_i^{d*}(u,s,\omega)dsdu}_{I_1} + \underbrace{\int_0^t \int_t^T \tilde{\sigma}_i^d(u,s,\omega)dsd\tilde{W}_i(u)}_{I_2}. \qquad \text{(D.13)}$$

It then follows that

$$\begin{aligned}
I &= \beta_f(t,T)\int_0^t \tilde{\sigma}_i^f(u,t,\omega)\int_u^t \tilde{\sigma}_i^f(u,v,\omega)dvdu + \beta_f(t,T)\int_0^t \tilde{\sigma}_i^f(u,t,\omega)d\tilde{W}_i(u) \\
&\quad + \beta_\lambda(t,T)\int_0^t \tilde{\sigma}_i^\lambda(u,t,\omega)\int_u^t \tilde{\sigma}_i^\lambda(u,v,\omega)dvdu + \beta_\lambda(t,T)\int_0^t \tilde{\sigma}_i^f(u,t,\omega)\int_u^t \tilde{\sigma}_i^\lambda(u,v,\omega)dvdu \\
&\quad + \beta_\lambda(t,T)\int_0^t \tilde{\sigma}_i^\lambda(u,t,\omega)\int_u^t \tilde{\sigma}_i^f(u,v,\omega)dvdu + \beta_\lambda(t,T)\int_0^t \tilde{\sigma}_i^\lambda(u,t,\omega)d\tilde{W}_i(u) \\
&\quad - \beta_\lambda(t,T)\int_0^t \tilde{\sigma}_i^\lambda(u,t,\omega)\int_u^t \tilde{\sigma}_i^f(u,v,\omega)dvdu + \beta_f(t,T)\int_0^t \tilde{\sigma}_i^\lambda(u,t,\omega)\int_u^t \tilde{\sigma}_i^f(u,v,\omega)dvdu \\
&\quad + \Big[\frac{1}{\kappa_\lambda}\beta_f(t,T) + \frac{1}{\kappa_f}\beta_\lambda(t,T) + \Big(\frac{1}{\kappa_f} + \frac{1}{\kappa_\lambda}\Big)\Big(\frac{1}{\kappa_f+\kappa_\lambda}\Big)\Big(1 - e^{-(\kappa_f+\kappa_\lambda)(T-t)}\Big)\Big] \\
&\quad \int_0^t \tilde{\sigma}_i^f(u,t,\omega)\tilde{\sigma}_i^\lambda(u,t,\omega)du + \frac{1}{2}\Big[\beta_f^2(t,T)\int_0^t \tilde{\sigma}_i^{2f}(u,t,\omega)du + \beta_\lambda^2(t,T)\int_0^t \tilde{\sigma}_i^{2\lambda}(u,t,\omega)du\Big],
\end{aligned}$$

$$\text{(D.14)}$$

which can be written as

$$I = \beta_f(t,T)[r(t,\omega) - f(0,t)] + \beta_\lambda(t,T)[\lambda(t,\omega) - \lambda(0,t)]$$
$$+ [\beta_f(t,T) - \beta_\lambda(t,T)] \int_0^t \tilde{\sigma}_i^\lambda(u,t,\omega) \int_u^t \tilde{\sigma}_i^f(u,v,\omega) dv du + A(t,T) \int_0^t \tilde{\sigma}_i^f(u,t,\omega)\tilde{\sigma}_i^\lambda(u,t,V) du$$
$$+ \frac{1}{2}\Big[\beta_f^2(t,T) \int_0^t \tilde{\sigma}_i^{2f}(u,t,\omega) du + \beta_\lambda^2(t,T) \int_0^t \tilde{\sigma}_i^{2\lambda}(u,t,\omega) du\Big], \tag{D.15}$$

where

$$A(t,T) = \Big[\frac{1}{\kappa_\lambda}\beta_f(t,T) + \frac{1}{\kappa_f}\beta_\lambda(t,T) + \Big(\frac{1}{\kappa_f} + \frac{1}{\kappa_\lambda}\Big)\Big(\frac{1}{\kappa_f + \kappa_\lambda}\Big)\Big(1 - e^{-(\kappa_f + \kappa_\lambda)(T-t)}\Big)\Big].$$

We can then write equation (D.5)

$$\bar{P}^d(t,T,\omega) = \frac{\bar{P}^d(0,T)}{\bar{P}^d(0,t)} \exp\Big[- \beta_f(t,T)[r(t,\omega) - f(0,t)] - \beta_\lambda(t,T)[\lambda(t,\omega) - \lambda(0,t)]$$
$$- [\beta_f(t,T) - \beta_\lambda(t,T)] \sum_{i=1}^3 \int_0^t \tilde{\sigma}_i^\lambda(u,t,V) \int_u^t \tilde{\sigma}_i^f(u,v,\omega) dv du$$
$$- A(t,T) \sum_{i=1}^3 \int_0^t \tilde{\sigma}_i^f(u,t,\omega)\tilde{\sigma}_i^\lambda(u,t,\omega) du$$
$$- \frac{1}{2}\Big[\beta_f^2(t,T) \sum_{i=1}^3 \int_0^t \tilde{\sigma}_i^{2f}(u,t,\omega) du + \beta_\lambda^2(t,T) \sum_{i=1}^3 \int_0^t \tilde{\sigma}_i^{2\lambda}(u,t,\omega) du\Big]\Big]. \tag{D.16}$$

We recall from Appendix C of Proposition 4.1 the definition of the state variables $\eta_1(t,\omega)$, $\eta_2(t,\omega)$, $\eta_3(t,\omega)$ and $S_3(t,\omega)$. Then, the equation for the pseudo bond above reduces to

$$\bar{P}^d(t,T,\omega) = \frac{\bar{P}^d(0,T)}{\bar{P}^d(0,t)} \exp\Big(- \frac{1}{2}\beta_f^2(t,T)\eta_1(t,\omega) - \frac{1}{2}\beta_\lambda^2(t,T)\eta_2(t,\omega) - A(t,T)\eta_3(t,\omega)$$
$$- \big[\beta_f(t,T) - \beta_\lambda(t,T)\big] S_3(t,\omega) - \beta_f(t,T)[r(t,\omega) - f(0,t)] - \beta_\lambda(t,T)[\lambda(t,\omega) - \lambda(0,t)]\Big). \tag{D.17}$$

We define the coefficients $\zeta_f(t,\omega) = r(t,\omega) - f(0,t)$, $\zeta_\lambda(t,\omega) = \lambda(t,\omega) - \lambda(0,t)$ and using the definition of the defaultable bond in terms of the pseudo bond given in equation (D.1), then the equation for the defaultable bond can be written as

$$P^d(t,T) = \mathcal{R}(t) \frac{\bar{P}^d(0,T)}{\bar{P}^d(0,t)} \exp\Big(- \frac{1}{2}\beta_f^2(t,T)\eta_1(t,\omega) - \frac{1}{2}\beta_\lambda^2(t,T)\eta_2(t,\omega) - A(t,T)\eta_3(t,\omega)$$
$$- \big[\beta_f(t,T) - \beta_\lambda(t,T)\big] S_3(t,\omega) - \beta_f(t,T)\zeta_f(t,\omega) - \beta_\lambda(t,T)\zeta_\lambda(t,\omega)\Big). \tag{D.18}$$

If in addition we define $D(t,T,\omega)$ to be such that

$$D(t,T,\omega) = -\ln\mathcal{R}(t) + \frac{1}{2}\beta_f^2(t,T)\eta_1(t,\omega) + \frac{1}{2}\beta_\lambda^2(t,T)\eta_2(t,\omega) + A(t,T)\eta_3(t,\omega) + \\ \big[\beta_f(t,T) + \beta_\lambda(t,T)\big]S_3(t,\omega),$$

then equation (D.18) yields equation 4.21 in Theorem 4.2.

Hence the proof.♦

E Proof of Theorem 7.1

We observe that equation (7.11) can be reduced to

$$\begin{aligned} P^d(t,T) = \frac{\bar{P}^d(0,T)}{\bar{P}^d(0,t)}\exp\bigg[& - ln\mathcal{R}(t) - \beta_f(t,T)[r(t,\omega) - f(0,t)] - \beta_\lambda(t,T)[\lambda(t,\omega) - \lambda(0,t)] \\ & - [\beta_f(t,T) - \beta_\lambda(t,T)]\sum_{i=1}^{3n}\int_0^t \tilde{\sigma}_i^\lambda(u,t,\omega)\int_u^t \tilde{\sigma}_i^f(u,v,\omega)dvdu \\ & - A(t,T)\sum_{i=1}^{3n}\int_0^t \tilde{\sigma}_i^f(u,t,\omega)\tilde{\sigma}_i^\lambda(u,t,\omega)du \\ & - \frac{1}{2}\Big[\beta_f^2(t,T)\sum_{i=1}^{3n}\int_0^t \tilde{\sigma}_i^{2f}(u,t,\omega)du + \beta_\lambda^2(t,T)\sum_{i=1}^{3n}\int_0^t \tilde{\sigma}_i^{2\lambda}(u,t,\omega)du\Big]\bigg]. \end{aligned} \tag{E.1}$$

Following a similar approach as in Appendix D yields the results of the theorem.♦