Introduction

0

- conditioning information definitions

  - let $\mathcal{F}_t$ be the set of info avail to investors at t

    let $\mathcal{F}_t^{\circ}$ be the set of uncond info: so $\mathcal{F}_t^{\circ} \subset \mathcal{F}_t$

  - let $\underline{r}_{t+1}$ be a $N \times 1$ set of basis returns

  - define

$X_{t+1} = \{ \underline{w}'_t \underline{r}_{t+1} \mid \underline{w}_t \in \mathcal{F}_t \}$ set of payoffs spanned by $\underline{r}_{t+1}$ using $\mathcal{F}_t$

$X^{\circ}_{t+1} = \{ \underline{w}' \underline{r}_{t+1} \mid \underline{w} \in \mathcal{F}^{\circ}_t \}$ set of payoffs spanned by $\underline{r}_{t+1}$ using $\mathcal{F}^{\circ}_t$

  - notice that $X^{\circ}_{t+1} \subset X_{t+1}$

  - X_{t+1} includes managed portfolios ie portfolios

whose weights in the N assets depend on the info available at t (so X_{t+1} not spanned by $\underline{r}_t$ using only $\mathcal{F}^{\circ}_t$)

  - 2 different expectation operators

conditional expectations $E[\cdot \mid \mathcal{F}_t] \equiv E_t[\cdot]$

unconditional expectations $E[\cdot \mid \mathcal{F}^{\circ}_t] = E[\ ]$

- understanding conditioning information

eg1 suppose $\mathcal{F}_t$ generated by d_t

$X_{t+1} = \{ \underline{w}'_t \underline{r}_{t+1} \mid \underline{w}_t = f(d_t) \quad f(\cdot) \text{ a well behaved fcn} \to \mathbb{R}^n \}$

so given d_t $w_t \in \mathbb{R}^N$ for each element of X_{t+1}

can think of $\mathcal{F}^{\circ}_t$ as being generated by a constant

$X^{\circ}_{t+1} = \{ \underline{w}' \underline{r}_{t+1} \mid \underline{w} = f \quad f \in \mathbb{R}^n \}$

①

- understanding expectations operators
  - conditional

$$E_t[y_{t+1}] = \underset{\substack{\mu_t^y \in \mathcal{F}_t \\ E[(\mu_t^y)^2] < \infty}}{\operatorname{argmin}} E[(y_{t+1} - \mu_t^y)^2 \mid \mathcal{F}^\circ]$$

so $E_t[y_{t+1}] \in \mathcal{F}_t$

$E_t[y_{t+1}]$ is the best guess of y_{t+1} given $\mathcal{F}_t$

- unconditional

$$E[y_{t+1}] = \underset{\substack{\mu^y \in \mathbb{R} \\ E[(\mu^y)^2] < \infty}}{\operatorname{argmin}} E[(y_{t+1} - \mu^y)^2 \mid \mathcal{F}^\circ]$$

so $E[y_{t+1}] \in \mathbb{R}$

$E[y_{t+1}]$ is the best guess of y_{t+1} given $\mathcal{F}_t^\circ$

eg1 cont suppose $\mathcal{F}_t$ generated by d_t

$$E_t[y_{t+1}] = \mu_t^y(d_t) \quad \& \quad \mu_t^y(\cdot) \text{ a well behaved fcn} \rightarrow \mathbb{R}$$

eg2 eg1 plus y_{t+1} & d_t are multivariate normal & $E d_t = 0$

$$E_t[y_{t+1}] = \mu^y + b\, d_t \quad \text{where} \quad b = \frac{cov[d_t, y_t]}{var[d_t]}$$

& $E[y_{t+1}] = \mu^y$

- 2 issues
  1. what is the payoff space?
  2. what is the expectations operator?

  - can calculate uncond expectations for X_{t+1}
  - can calculate cond " for X_{t+1}
  - can calculate uncond expectations for X_{t+1}°
  - can calculate cond " " X_{t+1}°

(2)

- useful results
  - law of iterated expectations $\mathcal{G}_t \subset \mathcal{F}_t \Rightarrow$

$$E[y_{t+1} \mid \mathcal{G}_t] = E[E[y_{t+1} \mid \mathcal{F}_t] \mid \mathcal{G}_t]$$

eg2 cont $\mathcal{F}_t^{\circ} \subset \mathcal{F}_t$

$$E_t[y_{t+1}] = \mu^y + b d_t$$

$$E[y_{t+1}] = \mu^y$$

$$E[E_t[y_{t+1}]] = E[\mu^y + b d_t] = \mu^y + b E[d_t] = \mu^y$$

note that the converse not hold

$$\mu^y = E_t[E[y_{t+1}]] \neq E_t[y_{t+1}] = \mu^y + b d_t$$

- $E[y_{t+1} z_t] = 0 \quad \forall z_t \in \mathcal{F}_t \Rightarrow E[y_{t+1} \mid \mathcal{F}_t] = 0$

eg1 cont $\mathcal{F}_t$ generated by d_t

$$E[y_{t+1} f(d_t)] = 0 \quad \forall f(\cdot) \text{ well behaved fcn}$$

$$\Rightarrow E[y_{t+1} \mid \mathcal{F}_t] = 0$$

- $\mathcal{G}_t \subset \mathcal{F}_t$

$$\text{cov}[x_{t+1}, y_{t+1} \mid \mathcal{G}_t] = E[\text{cov}_t[x_{t+1}, y_{t+1}] \mid \mathcal{G}_t] + \text{cov}[E_t[x_{t+1}], E_t[y_{t+1}] \mid \mathcal{G}_t]$$

pf

$$\text{cov}[x_{t+1}, y_{t+1}] = E[x_{t+1} y_{t+1}] - E[x_{t+1}]E[y_{t+1}]$$

$$= E\big[E_t[x_{t+1} y_{t+1}] - E_t[x_{t+1}]E_t[y_{t+1}]\big] + E\big[E_t[x_{t+1}]E_t[y_{t+1}]\big] - E\big[E_t[x_{t+1}]\big]E\big[E_t[y_{t+1}]\big]$$

③

- let $L^2_{t+1} = \{ y_{t+1} : y_{t+1} \in \mathcal{F}_{t+1},\ E[y^2_{t+1} \mid \mathcal{F}_t] < \infty \}$

Theorem: H_{t+1} closed linear subspace of L^2_{t+1}. For $y_{t+1} \in L^2_{t+1}$

$\text{proj}_t(y_{t+1} \mid H_{t+1})$ exists & satisfies

$$\underset{h_{t+1} \in H_{t+1}}{\text{argmin}}\ E_t[(y_{t+1} - h_{t+1})^2] = h^0_{t+1}$$

& $h^0_{t+1} \in H_{t+1} \ni$ $\quad E_t[(y_{t+1} - h^0_{t+1})\, h_{t+1}] = 0 \quad \forall\, h_{t+1} \in H_{t+1}$

RRT: Let H_{t+1} be a closed linear subspace of L^2_{t+1}

& $\pi_t(\ )$ a contin lin fcnal on H_{t+1}.

Then $\exists$ a unique element $h^{\pi_t}_{t+1} \in H_{t+1} \ni$

$$E_t[h_{t+1}\, h^{\pi}_{t+1}] = \pi_t(h_{t+1}) \quad \forall\, h_{t+1} \in H_{t+1}$$

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Law of One Price, No Arbitrage & Stochastic Discount Factors

- payoffs
  - $\rightarrow$ let $\tilde{X}_{t+1}$ be the set of asset payoffs at $t+1$
- prices
  - $\rightarrow$ let $p_t(x_{t+1})$ be the time-t price associated with payoff $x_{t+1} \in \tilde{X}_{t+1}$; so $p_t(\cdot) \in \mathcal{F}_t$
  - $\rightarrow$ p_t not uniquely determined without additional assump
  - $\rightarrow$ so p_t is a correspondence & not nec a fcn
- defn of no arbitrage
  - $\rightarrow$ a payoff space $\tilde{X}_{t+1}$ + a pricing correspondence $p_t(\cdot)$ have no arb op if for every payoff $x_{t+1} \in \tilde{X}_{t+1}$ $\ni x \geq 0$ with prob 1 wrt $\mathcal{F}_t$ & $\left(E[(x_{t+1})^2 \mid \mathcal{F}_t]\right)^{1/2} > 0$, $p > 0$ $\forall p \in p_t(x_{t+1})$

ie if every payoff $x_{t+1} \in \tilde{X}_{t+1} \ni x < 0$ with zero prob & $x_{t+1} > 0$ with pos prob (using $\mathcal{F}_t$) has $p_t(x_{t+1}) > 0$

- defn of law of one price
  - $\rightarrow$ A2: $p_t(\cdot)$ is a linear functional so if $x^1 = x^2 \in \tilde{X}_{t+1}$ then $p_t(x^1) = p_t(x^2)$
  - $\rightarrow$ A2': if x^1 & $x^2 \in \tilde{X}_{t+1}$ & w^1 & $w^2 \in \mathcal{F}_t$ then $p_t(w^1 x^1 + w^2 x^2) = w^1 p_t(x^1) + w^2 p_t(x^2)$ (1)

- setting

→ $L^2_{t+1} = \{ y_{t+1} \; y_{t+1} \in \mathcal{F}_{t+1}, \; E[y^2_{t+1} | \mathcal{F}_t] < \infty \}$ +

- can show L^2_{t+1} is a Hilbert space
- L^2_{t+1} can be infinite dimensional

→ A1: let $\tilde{X}_{t+1}$ be $X_t \equiv \{ x_{t+1} | x_{t+1} = \sum_{i \in I} w^i_t x^i_{t+1} \; w^i_t \in \mathcal{F}_t \}$

which is the linear span of x^i_{t+1} $i \in I$ $x^i_{t+1} \in L^2_{t+1}$

(if I does not have a finite # of elements may need to take the closure of the span)

- closure of X_{t+1} is a closed linear subspace of L^2_{t+1}

→ let $p_t(\cdot)$ be the price correspondence for X_{t+1}

(again if I does not have a finite # elements need to augment the set of prices in a natural way)

→ suppose $\exists \; x^0_{t+1} \in X_{t+1} \ni x^0_{t+1} \geq 0$ +

$\| x^0_{t+1} \|_t \equiv \left(E[(x^0_{t+1})^2 | \mathcal{F}_t] \right)^{1/2} > 0$

*

- relation between PNA + LOOP

→ Prop: PNA ⟹ LOOP

Proof (finite # of assets)

- let N be the number of assets

$\underset{N \times 1}{\underline{P}_t}$ prices $\qquad$ $\underset{N \times 1}{\underline{x}_{t+1}}$ payoffs

2

- prove by contradiction given PNA holds for x^0_{t+1}
  no LOOP $\Rightarrow$ no PNA
- take $\hat{x} \in X_{t+1}$
  suppose $\hat{p}, \hat{p}^* \in p_{t+1}(\hat{x})$ $\quad \hat{p} < \hat{p}^*$
  so $\exists$ $\underline{\hat{\omega}}$ & $\underline{\hat{\omega}}^* \in \mathcal{G}_t$ $\ni$
  $\hat{x} = \underline{\hat{\omega}}' \underline{x}_{t+1} = (\underline{\hat{\omega}}^*)' \underline{x}_{t+1}$
  & $\underline{\hat{\omega}}' \underline{p}_t = \hat{p}$ & $(\underline{\hat{\omega}}^*)' \underline{p}_t = \hat{p}^*$
- since PNA holds for x^0_{t+1}, then for $\underline{\omega}^0$
  $\ni$ $x^0_{t+1} = (\underline{\omega}^0)' \underline{x}$, $p^0_t = (\underline{\omega}^0)' \underline{p}_t > 0$
- consider a portfolio with weight vector

$$\underbrace{\frac{\hat{p}^* - \hat{p}}{p^0_t} \underline{\omega}^0}_{\frac{\hat{p}^* - \hat{p}}{p^0_t} \text{ in } x^0_{t+1}} + \underbrace{\underline{\hat{\omega}}}_{\substack{\text{buy} \\ \text{cheap} \\ \hat{x}_{t+1}}} - \underbrace{\underline{\hat{\omega}}^*}_{\substack{\text{sell} \\ \text{expensive} \\ \hat{x}_{t+1}}}$$

- cost of this portfolio

$$\left(\frac{\hat{p}^* - \hat{p}}{p^0_t} \right) p^0_t + \hat{p} - \hat{p}^* = 0$$

- portfolio payoff

$$\frac{\hat{p}^* - \hat{p}}{p^0_t} x^0_{t+1} + \hat{x} - \hat{x} \geq 0 \quad \& \quad \left\| \frac{\hat{p}^* - \hat{p}}{p^0_t} x^0_{t+1} \right\| > 0$$

which contradicts PNA

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→ can also show that PNA ⇒ price functional is continuous

- defn of a stochastic discount factor sdf

→ $m_{t+1} \in L^2_{t+1}$ is a sdf for $\tilde{X}_{t+1}$ wrt $\hat{\mathcal{F}}_t \subset \mathcal{F}_t$

iff

$$E[m_{t+1} x_{t+1} | \hat{\mathcal{F}}_t] = E[p_t(x_{t+1}) | \hat{\mathcal{F}}_t]$$

$$\forall x_{t+1} \in \tilde{X}_{t+1}$$

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- implications of LOOP

→ setting

· (*)

→ LOOP implies the existence of a sdf

Prop: A2 (& a contin price functional)

$\Rightarrow \exists$ a unique payoff $m^*_{t+1} \in X_{t+1}$

$\ni \; p_t(x_{t+1}) = E_t[m^*_{t+1} x_{t+1}] \;\; \forall x_{t+1} \in X_{t+1}$

Pf (finite # of assets)

· suppose as before X_{t+1} generated by a $N \times 1$ vector of basic payoffs $\underline{x}_{t+1}$ with prices $\underline{p}_t$

· choose

$$\underset{1\times 1}{m^*_{t+1}} = \underset{1\times N}{\underline{p}_t'} \underset{N\times N}{\left(E_t[\underline{x}_{t+1}\underline{x}'_{t+1}]\right)^{-1}} \underset{N\times 1}{\underline{x}_{t+1}}$$

· this m^*_{t+1} satisfies

$$\underline{p}'_t = E_t[m^*_{t+1} \underline{x}'_{t+1}]$$

&

$$\underline{p}'_t \underline{c} = E_t[m^*_{t+1} (\underline{x}'_{t+1} \underline{c})] \quad \forall \underline{c} \in \mathcal{F}_t$$

(infinite # of assets)

RRT

- implications of the existence of a sdf

→ setting

(*)

Theorem: sdf exists for $X_{t+1} \Rightarrow$ LOOP holds

pf: immediate

⑤

- implication of PNA

→ setting. (*)

→ PNA implies the existence of a strictly pos sdf

Prop: PNA $\Rightarrow \exists\, m_{t+1} \in L^2_{t+1} \ni$ prob $m_{t+1} > 0$ is 1 & $p_t(x_{t+1}) = E_t[m_{t+1}\, x_{t+1}]\ \forall\, x_{t+1} \in X_{t+1}$

Pf: omitted

- implication of existence of a strictly pos sdf

→ setting. (*)

→ Prop: $m_{t+1} \in L^2_{t+1}$ exists $\ni$ prob $m_{t+1} > 0$ is 1 & $p_t(x_{t+1}) = E_t[m_{t+1}\, x_{t+1}]\ \forall\, x_{t+1} \in X_{t+1}$

$\Rightarrow$ PNA

Proof (heuristic): . consider $x \in X_{t+1} \ni x \geq 0$ with prob 1 & $(E_t[x^2])^{1/2} > 0$

. now

$$p_t(x) = E_t[\underbrace{m_{t+1}}_{\text{strictly pos}}\ \underbrace{x}_{\text{pos with pos prob}}] > 0$$

so PNA holds

⑥

- complete vs incomplete markets

→ consider a one period setting with a finite number of states S & $\sigma_t = \sigma_t^0$

1. if N, the number of assets is less than S the number of states, the market is incomplete
2. if $N = S$ the market is complete
3. in particular, let $\underline{x}_{t+1}$ be the $N \times S$ vector of basis assets $\underline{x}_{t+1} = \begin{bmatrix} \underline{x}^1_{t+1} \\ \vdots \\ \underline{x}^N_{t+1} \end{bmatrix}$ $\underline{x}^i_{t+1} \in \mathbb{R}^S$ $i = 1, 2, \ldots, N$

then • if $N < S$ then the payoff space

$$X_{t+1} = \{ \underline{w}\, \underline{x}_{t+1} : w \in \mathbb{R}^N \} \subsetneq \mathbb{R}^S$$

• if $N = S$ then $X_{t+1} = \mathbb{R}^S$ & the market is complete

- summary of implications of PNA
  - complete markets
    1) one m_{t+1} that satisfies Euler eqn (& that is m^*_{t+1})
    2) the one m_{t+1} is strictly pos
    3) The one m_{t+1} is m^*_{t+1} (which must be strictly pos)
  - incomplete markets
    1) many m_{t+1} that satisfy Euler eqn
    2) at least one is strictly positive
    3) m^*_{t+1} need not be strictly positive
    4) relation m_{t+1} & m^*_{t+1}

       can decompose any m_{t+1}

       $$m_{t+1} = m^*_{t+1} + e_{t+1}$$

       where $E_t[e_{t+1}\, x_{t+1}] = 0 \quad \forall\, x_{t+1} \in X_{t+1}$

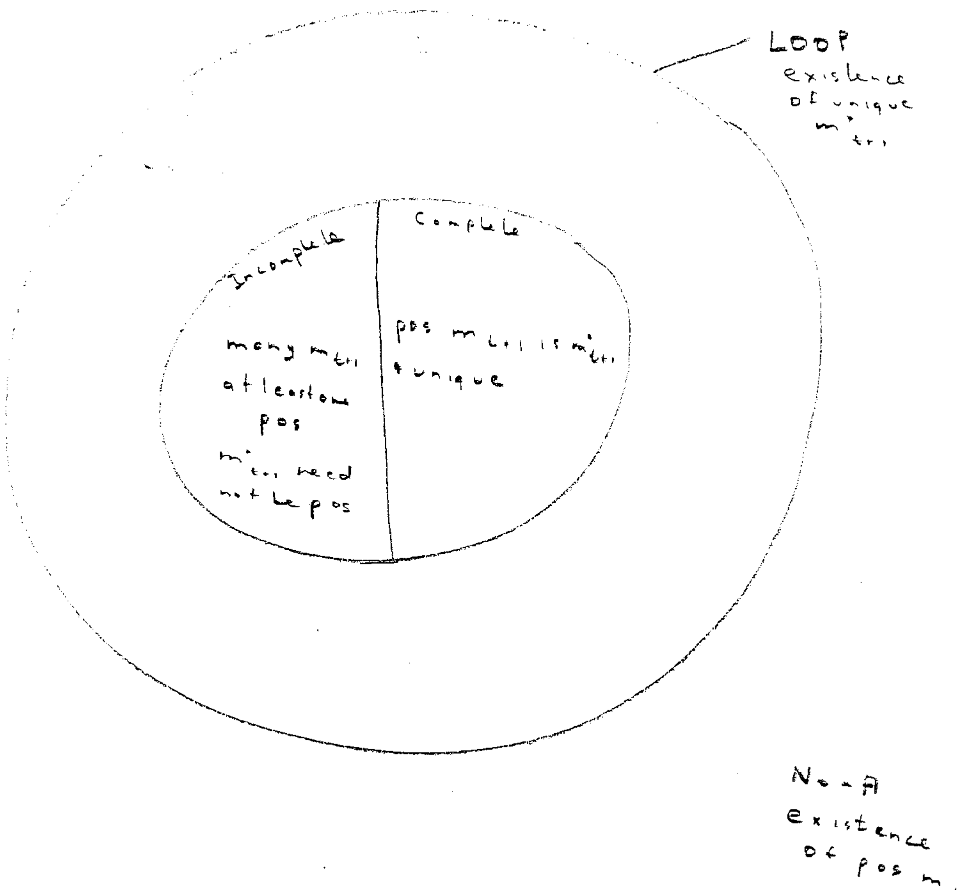
    then

    $$E_t[m_{t+1}\, x_{t+1}] = E_t[m^*_{t+1}\, x_{t+1}] + E_t[e_{t+1} x_{t+1}] = E_t[m^*_{t+1} x_{t+1}]$$

so obtain m^*_{t+1} by regressing m_{t+1} on $\underline{x}$ where $\underline{x}$ is a basis for X_{t+1}

(8)

$$\begin{array}{ccc} \text{No A} & \Longrightarrow & \text{LOOP} \\ \Updownarrow & & \Updownarrow \\ \text{existence of a positive } m_{t+1} & \Longrightarrow & \text{existence of } m_{t+1} \end{array}$$



- implications of finding an sdf wrt $\mathcal{F}_t$

→ suppose m_{t+1} constitutes an sdf for $\tilde{X}_{t+1}$ wrt $\mathcal{F}_t$ ie

$$E_t[\, m_{t+1} \; x_{t+1} \,] = p_t(x_{t+1})$$

$$\forall \; x_{t+1} \in \tilde{X}_{t+1}$$

→ then

1) $$1 = E_t[\, m_{t+1} \; r^f_{t+1} \,] \Rightarrow r^f_{t+1} = \frac{1}{E_t[m_{t+1}]}$$

2) let $\tilde{R}_{t+1} = \left\{ r_{t+1} \in \tilde{X}_{t+1} \;\middle|\; p_t(r_{t+1}) = 1 \right\}$

+ suppose $r^f_{t+1} \in \tilde{X}_{t+1}$ then

$$1 = E_t[m_{t+1} \; r^i_{t+1}] \qquad \forall \; r^i_{t+1} \in \tilde{R}_{t+1}$$

$$1 = \mathrm{cov}_t[m_{t+1}, r^i_{t+1}] + E_t[m_{t+1}] \, E_t[r^i_{t+1}]$$

$$E_t[r^i_{t+1}] = \frac{1}{E_t[m_{t+1}]} - \frac{1}{E_t[m_{t+1}]} \mathrm{cov}_t[m_{t+1}, r^i_{t+1}]$$

$$= r^f_{t+1} + \underbrace{\left\{ - R^f_{t+1} \; \mathrm{var}_t[m_{t+1}] \right\}}_{\lambda_m} \underbrace{\left\{ \frac{\mathrm{cov}_t[m_{t+1}, r^i_{t+1}]}{\mathrm{var}_t[m_{t+1}]} \right\}}_{\beta_{i,m}}$$

⑩

3) same as 2) $r^i_{t+1} \in \tilde{R}_{t+1}$

$$1 = E_t[m_{t+1} r^i_{t+1}] \quad \& \quad 1 = E_t[m_{t+1} r^f_{t+1}]$$

$$0 = E_t[m_{t+1}\{r^i_{t+1} - r^f_{t+1}\}]$$

$$0 = \mathrm{cov}_t[m_{t+1}\{r^i_{t+1} - r^f_{t+1}\}] + E_t[m_{t+1}]E_t[r^i_{t+1} - r^f_{t+1}]$$

$$\frac{E_t r^i_{t+1} - r^f_{t+1}}{r^f_{t+1}} = -\rho_t[m_{t+1}, r^i_{t+1}]\,\sigma_t[m_{t+1}]\,\sigma_t[r^i_{t+1}]$$

$$\Rightarrow \left| \left\{ \frac{E_t r^i_{t+1} - r^f_{t+1}}{\sigma_t[r^i_{t+1}]} \right\} \frac{1}{r^f_{t+1}} \right| \leq \sigma_t[m_{t+1}]$$

holds for any $r^i_{t+1} \in \tilde{R}_{t+1}$

so if $\underline{r}$ forms a return basis (wrt $\mathcal{F}_t$) for $\tilde{x}_{t+1}$ & $E_t[\underline{r}] = \underline{\mu}_t$ & $\mathrm{cov}_t[\underline{r}] = V_t$

$$\left| \underbrace{\left((\underline{\mu}_t - r^f_{t+1}\underline{i})' V_t^{-1} (\underline{\mu}_t - r^f_{t+1}\underline{i}) \right)^{\frac{1}{2}}}_{\substack{\text{max} \\ \text{Sharpe measure} \\ \text{H-J bound}}} \frac{1}{R^f_{t+1}} \right| \leq \sigma_t[m_{t+1}]$$

- consumption-based pricing
  - → consider the following problem

$$\max \; E\left[\sum_{\tau=0}^{\infty} \beta^{\tau} u(c_\tau)\right] \qquad \text{w.r.t. } \{c_\tau\}_{\tau=0}^{\infty} \; \{\underline{\alpha}_\tau\}_{\tau=0}^{\infty}$$

s.t. W_0

$$W_{\tau+1} = (W_\tau - c_\tau + y_\tau)(\underline{\alpha}'_\tau(\underline{r}_{\tau+1} - r^f_{\tau+1}\underline{\iota}) + r^f_{\tau+1})$$

$$\tau = 0, 1, \ldots$$

$$\& \quad \{c_\tau, \underline{\alpha}_\tau\} \in \mathcal{F}_\tau \qquad \tau = 0, 1$$

→ suppose $\{c^*_\tau \; \underline{\alpha}^*_\tau\}_{\tau=0}^{\infty}$ is the soln

→ consider the following perturbation

- reduce c^*_t by δz_t $\qquad z_t \in \mathcal{F}_t$
- invest δz_t in asset i
- increase c^*_{t+1} by $\delta z_t r^i_{t+1}$
- leave all other c^*_τ s the same
- choose δ
- since along optimal path $\delta^* = 0$ solves the following problem

$$\max_{\delta} \; E[\beta^t u(c^*_t - \delta z_t) + \beta^{t+1} u(c^*_t + \delta z_t r^i_{t+1})]$$

foc

$$E[-\beta^t u'(c^*_t - \delta^* z_t) z_t + \beta^{t+1} u'(c^*_{t+1} + \delta^* z_t r^i_{t+1}) z_t r^i_{t+1}] = 0$$

set $\delta^* = 0$

$$E[z_t \{u'(c^*_t) - \beta u'(c^*_{t+1}) R^i_{t+1}\}] = 0$$

(12)

holds for any $z_t \in \mathcal{F}_t$

& so

$$E\left[\, u'(c_t^*) - \beta\, u'(c_{t+1}^*)\, r_{t+1}^i \,\middle|\, \mathcal{F}_t \right] = 0$$

&

$$u'(c_t^*) = \beta\, E_t\left[u'(c_{t+1}^*)\, r_{t+1}^i\right]$$

& so

$$1 = E_t\left[\left\{\beta \frac{u'(c_{t+1}^*)}{u'(c_t^*)}\right\} r_{t+1}^i\right] \qquad \text{returns}$$

also

$$p_t^i = E_t\left[\left\{\ \text{''}\ \right\} p_{t+1}^i\right] \qquad \text{payoffs}$$

&

$$z_t\, p_t^i = E_t\left[\left\{\ \text{''}\ \right\} z_t\, p_{t+1}^i\right] \qquad \text{scaled payoffs}$$

for any $z_t \in \mathcal{F}_t$

$$\& \quad 0 = E_t\left[\left\{\ \text{''}\ \right\} \left(r_{t+1}^i - r_{t+1}^j\right)\right] \qquad \text{zero investment portfolio}$$

more gen

$$z_t^f + \underline{z}_t'\, \underline{i} = E_t\left[\left\{\ \text{''}\ \right\}\left(z_t^f\, r_{t+1}^f + \underline{z}_t'\, \underline{r}_{t+1}\right)\right]$$

$\forall\ \underline{z}_t, z_t^f \in \mathcal{F}_t$

(13)

– application to consumption based model

$$m_{t+1} = \frac{\beta\, u'(c_{t+1})}{u'(c_t)}$$

1) $$\frac{1}{r^f_{t+1}} = E_t\left[\frac{\beta\, u'(c_{t+1})}{u'(c_t)}\right]$$

2) $$E_t\, r^i_{t+1} = r^f_{t+1} - r^f_{t+1}\, \mathrm{cov}_t\left[\frac{\beta\, u'(c_{t+1})}{u'(c_t)}, r^i_{t+1}\right]$$

so $$\mathrm{cov}_t\left[u'(c_{t+1}), r^i_{t+1}\right] \uparrow \qquad E_t\, r^i_{t+1} \downarrow$$

$$\mathrm{cov}_t\left[c_{t+1}, r^i_{t+1}\right] \downarrow \qquad E_t\, r^i_{t+1} \downarrow$$

asset
more likely
to pay off
in bad states

3) $$\left| (\mu_t - r^f_{t+1}\,\underline{\iota})' V_t^{-1} (\mu_t - r^f_{t+1}\,\underline{\iota}) \frac{1}{r^f_{t+1}} \right| \le \sigma_t\left[\frac{\beta\, u'(c_{t+1})}{u'(c_t)}\right]$$

Stochastic Discount Factors & the

$\mathcal{F}_t$ - Conditional Minimum-Variance Frontier

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- setting
  - $\cdot$ * + NA
- define X_{t+1} as before
- relation m^* + the $\mathcal{F}_t$-conditional

minimum variance frontier MVF

$\cdot$ define $R_{t+1} \equiv \{ r_{t+1} \in X_{t+1} \mid p_t(r_{t+1}) = 1 \}$

$\cdot$ know $\exists$ a unique $m^*_{t+1} \in X_{t+1}$

$\ni E_t[m^*_{t+1} \, x_{t+1}] = p_t(x_{t+1}) \quad \forall x_{t+1} \in X_{t+1}$

$\cdot$ so $E_t[m^*_{t+1} \, m^*_{t+1}] = p_t(m_{t+1})$ since $m^*_{t+1} \in X_{t+1}$

$\cdot$ so defining $r^*_{t+1} = \dfrac{m^*_{t+1}}{E_t[m^{*2}_{t+1}]}$, know

$r^*_{t+1} \in R_{t+1}$ since $E_t[m^*_{t+1} \, r^*_{t+1}] = E_t\left[m^*_{t+1} \dfrac{m^*_{t+1}}{E_t[m^{*2}_{t+1}]}\right] = 1$

①

Theorem: r^*_{t+1} is the minimum second moment return (wrt $\mathcal{G}_t$)

Pf second moment (uncentered) for $x_{t+1} \in X_{t+1}$ is $E_t[x^2_{t+1}]$. Know $r^*_{t+1} = \frac{m^*_{t+1}}{E_t[m^{*2}_{t+1}]}$

So $E_t[r^{*2}_{t+1}] = E_t[m^{*2}_{t+1}] / (E_t[m^{*2}_{t+1}])^2 = \frac{1}{E_t[m^{*2}_{t+1}]}$ & $m^*_{t+1} = \frac{r^*_{t+1}}{E_t[r^{*2}_{t+1}]}$

For $r_{t+1} \in R_{t+1}$, can write
$r_{t+1} = r^*_{t+1} + z_{t+1}$ where $z_{t+1} = r_{t+1} - r^*_{t+1}$

Now $E_t\, m^*_{t+1} r_{t+1} = 1$ so $E_t\, r^* r_{t+1} = E_t[r^{*2}_{t+1}]$

since $m^*_{t+1} = \frac{r^*_{t+1}}{E_t[r^{*2}_{t+1}]}$

Thus $E_t\, r^*_{t+1} r_{t+1} = E_t[r^*_{t+1}(r^*_{t+1} + z_{t+1})]$

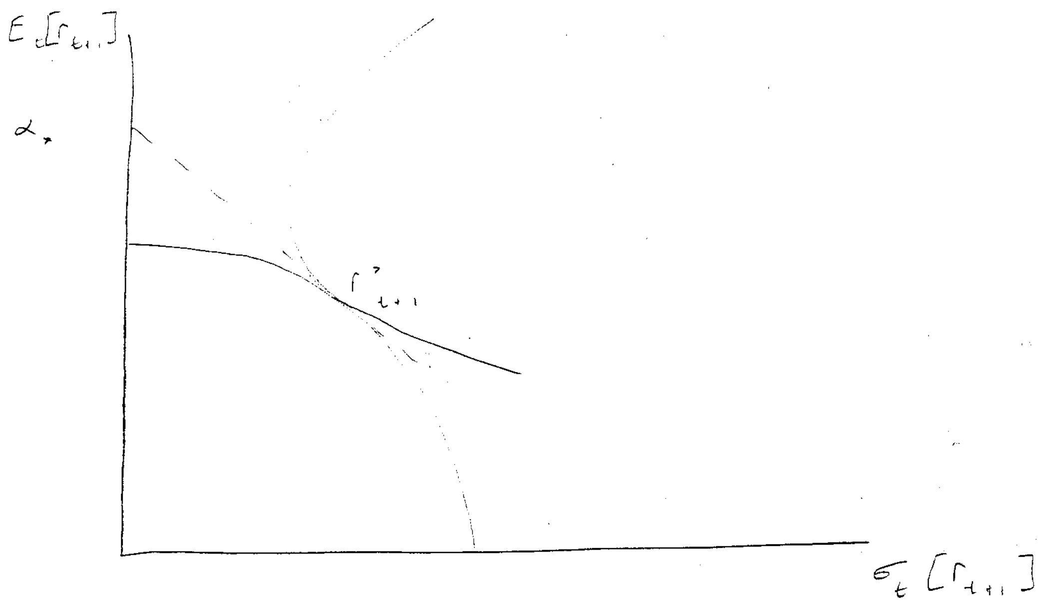
$\Longrightarrow E_t[r^*_{t+1} z_{t+1}] = 0$

So $E_t[r^2_{t+1}] = E_t[r^{*2}_{t+1}] + E_t[z^2_{t+1}] \geq E_t[r^{*2}_{t+1}]$
$\forall\, r_{t+1} \in R_{t+1}$

so r^*_{t+1} is min second moment return

Corollary: Since $\sigma^2_t[r_{t+1}] + (E_t[r_{t+1}])^2 = E_t[r^2_{t+1}]$
it follows r^*_{t+1} is on the $\mathcal{G}_t$-cond. MVF

②



- relation r^{mv} on MVF, an m_{t+1} which works & $\beta_{i,mv}$ pricing

Theorem: r^{mv}_{t+1} on MVF for R_{t+1} $\iff$ $m_{t+1} = a + b r^{mv}_{t+1}$, $a, b \in \mathcal{G}_t$ is a sdf w.r.t. $\mathcal{G}_t$ for X_{t+1}

$\iff$ $E_t[r^i_{t+1}] - \alpha_{mv} = \beta_{i,mv} \lambda_{mv} \quad \forall r^i_{t+1} \in R_{t+1}$

where $\beta_{i,mv} = \dfrac{cov_t[r^i_{t+1}, r^{mv}_{t+1}]}{var_t[r^{mv}_{t+1}]}$ & $\lambda_{mv} = E_t[r^{mv}_{t+1}] - \alpha_{mv}$

& $\alpha_{mv} = \dfrac{1}{E_t[m_{t+1}]}$

define $Z_{t+1} \equiv \{ z_{t+1} \in X_{t+1} : p_t(z_{t+1}) = 0 \}$

Lemma: $r^i_{t+1} \in R_{t+1}$ can be expressed as

$$r^i_{t+1} = r^*_{t+1} + w^i z^*_{t+1} + n^i_{t+1}$$

where $E_t[z^*_{t+1} z^i_{t+1}] = E_t[z^i_{t+1}] \quad \forall z^i_{t+1} \in Z_t$

& $w^i \in \mathcal{G}_t$ is chosen so that $E_t\, z^*_{t+1} n^i_{t+1} = 0$

moreover,

i $z^*_{t+1} = proj(1 \mid Z_{t+1})$

& ii $E_t[r^*_{t+1} z^*_{t+1}] = E_t[r^*_{t+1} n^i_{t+1}] = 0$

iii $E_t[n^i_{t+1}] = 0$

3

Pf Lemma

1) <u>Existence of z^-_{t+1}</u>

Use RRT & linear functional $E_t[\cdot]$ to establish the existence of z^-_{t+1}

4

2) Result ii

$$E_t[r^*_{t+1} r^i_{t+1}] = E_t[r^{*2}_{t+1}]$$

$$= E_t[r^*_{t+1}(r^*_{t+1} + w^i z^*_{t+1} + n^i_{t+1})]$$

$$\Rightarrow E_t[r^*_{t+1} z^*_{t+1}] = E_t[r^*_{t+1} n^i_{t+1}] = 0$$

since $z^*_{t+1} \in Z_{t+1}$ implies $E_t[r^*_{t+1} z^*_{t+1}] = 0$

3) Result i

Now $E_t[1 \; z^i_{t+1}] = E_t[(proj(1|Z_{t+1}) + e) z^i_{t+1}]$

$$= E_t[proj(1|Z_{t+1}) z^i_{t+1}] \quad \forall z^i_{t+1} \in Z_{t+1}$$

$$= E_t[z^*_{t+1} z^i_{t+1}].$$

4) Result iii

$$E_t[z^*_{t+1} z^i_{t+1}] = E_t z^*_{t+1}(w^i z^*_{t+1} + n^i_{t+1})$$

$$= w^i E_t z^*_{t+1}$$

$$E_t[z^i_{t+1}] = E_t[w^i z^*_{t+1} + n^i_{t+1}]$$

$$= w^i E_t z^*_{t+1} + E_t[n^i_{t+1}]$$

So $E_t n^i_{t+1} = 0$

Lemma: r^{mv}_{t+1} is on MVF iff $r^{mv}_{t+1} = r^*_{t+1} + w z^*_{t+1}$

for some $w \in \mathcal{F}_t$

Pf Lemma:

For any $r^i_{t+1} \in R_{t+1}$ $\quad r^i_{t+1} = r^*_{t+1} + w^i z^*_{t+1} + n^i_{t+1}$

$$E_t r^i_{t+1} = E_t r^*_{t+1} + w^i E_t z^*_{t+1}$$

$$\& \; \sigma_t^2(r^i_t) = \sigma^2_t(r^*_{t+1} + w^i z^*_{t+1}) + \sigma^2_t(n^i_{t+1})$$

For each mean there is a unique w^i. Returns with $n^i_{t+1} = 0$ min var for each mean eg suppose $r^f_{t+1} \in R_{t+1}$ Know r^f_{t+1} is on MVF

So $r^f_{t+1} = r^*_{t+1} + w\, z^*_{t+1}$ for some $w \in \mathcal{G}_t$

$$z^*_{t+1} = (r^f_{t+1} - r^*_{t+1})\frac{1}{w}$$

Know $E_t[z^*_{t+1}\, z^i_{t+1}] = E_t\left[(r^f_{t+1} - r^*_{t+1})\frac{1}{w}\, z^i_{t+1}\right] = E_t[z^i_{t+1}]$

$\Rightarrow E_t\, r^f_{t+1}\frac{1}{w} z^i_{t+1} = E_t\, z^i_{t+1}$ since $\frac{1}{w}E_t[r^*_{t+1}\, z^i_{t+1}] = 0$

$\Rightarrow \left(r^f_{t+1}\frac{1}{w}\right) E_t\, z^i_{t+1} = E_t\, z^i_{t+1} \quad \forall\, z^i_{t+1} \in Z_{t+1}$

$\Rightarrow w = r^f_{t+1}$

Pf Theorem Take an arbitrary $r_{t+1} \in R_{t+1}$

& set $m_{t+1} = a + b r^i_{t+1} = a + b(r^*_{t+1} + w\, z^*_{t+1} + n_{t+1})$

Will show m_{t+1} is a valid sdf for X_{t+1}

iff $n_{t+1} = 0$ for case $r^f_{t+1} \notin X_{t+1}$

Now m_{t+1} must at least price r^*_{t+1} & z^*_{t+1}

So $1 = E_t[m_{t+1}\, r^*_{t+1}] = a E_t[r^*_{t+1}] + b E_t[r^{*2}_{t+1}]$

$0 = E_t[m_{t+1}\, z^*_{t+1}] = a E_t[z^*_{t+1}] + b w\, E_t[z^{*2}_{t+1}]$

$= a E_t[z^*_{t+1}] + b w\, E_t[z^*_{t+1}]$

$\Rightarrow 0 = a + bw$

solving for a & b

$$a = \frac{w}{(w E_t r^*_{t+1} - E_t(r^{*2}_{t+1}))} \qquad b = -\frac{a}{w}$$

& so if this m_{t+1} works it must be

(6) $$m_{t+1} = \frac{w - (r^*_{t+1} + w z^*_{t+1} + n)}{w E_t r^*_{t+1} - E_t[r^{*2}_{t+1}]}$$

Now to price an arbitrary $x^i \in X_{t+1}$

note that $x^i_{t+1} = p_t(x^i_{t+1})\, r^*_{t+1} + w^i z_{t+1} + n^i_{t+1}$

so

$$E_t[m_{t+1}\, x^i_{t+1}] = E_t\left[\frac{(w - (r^*_{t+1} + w z^*_{t+1} + n))\,(p_t(x^i_{t+1})\, r^*_{t+1} + w^i z^*_{t+1} + n^i_{t+1})}{w E_t r^*_{t+1} - E_t[r^{*2}_{t+1}]} \right]$$

$\Longrightarrow$

$$E_t[m_{t+1}\, x^i_{t+1}] = p_t(x^i_{t+1}) - \frac{E_t[n\, n^i]}{w\, E r^* - E r^{*2}}$$

note

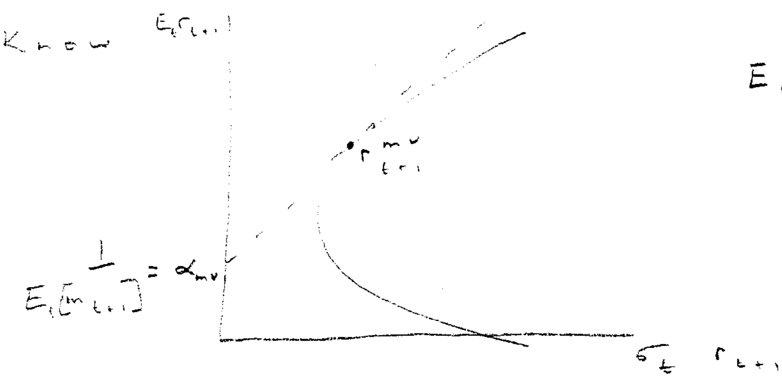
$$E_t[(a + b\, r^{mv}_{t+1})\, x^i_{t+1}] = p_t(x^i_{t+1}) \quad \forall\, x^i_{t+1} \in X_{t+1}$$

$$E_t[(a + b\, r^{mv}_{t+1})]\, E_t r^i_{t+1} + \mathrm{cov}_t[b\, r^{mv}_{t+1}, r^i_{t+1}] = 1$$

$$\Longleftrightarrow \quad E_t r^i_{t+1} = \underbrace{\frac{1}{E_t m_t}}_{\alpha_{mv}} - \frac{b}{E_t m_t}\, \mathrm{cov}_t[r^{mv}_{t+1}, r^i_{t+1}]$$

- intuition

Know

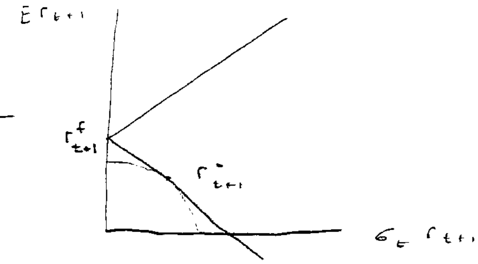


$$E_t r^i_{t+1} = \alpha_{mv} + \lambda_{mv}\, \beta_{i,mv}$$

(7)

• if $r^f_{t+1} \in R_{t+1}$ then

1) $m_{t+1} = a + b\, r^{mv}_{t+1} = m^*_{t+1}$ $\forall\ r^{mv}_{t+1}$ on MVF except r^f_{t+1}



2) $\frac{1}{E_t[m^*_{t+1}]} = \frac{E_t[r^{*2}_{t+1}]}{E_t[r^*_{t+1}]} = r^f_{t+1} = E_t r^{gmv}_{t+1} = \frac{E_t r^*_{t+1}}{1 - E_t z^*_{t+1}}$

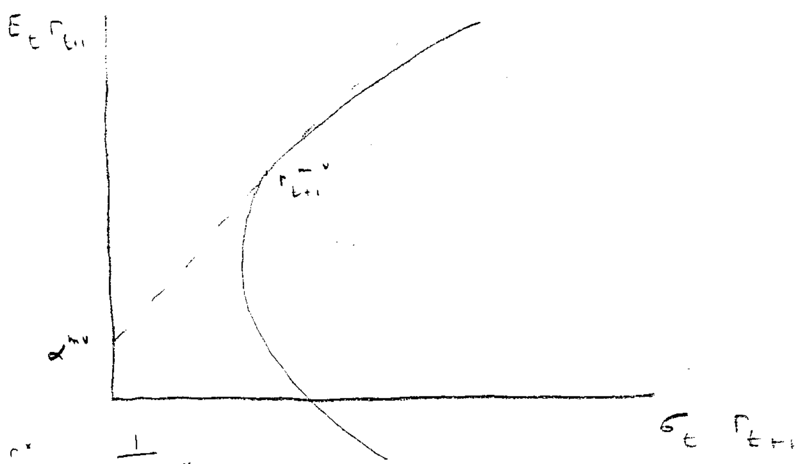
3) $w = \frac{E_t[r^{*2}_{t+1}]}{E_t[r^*_{t+1}]} = w^f = w^{gmv} = r^f_{t+1}$ corresponds to $m_{t+1} =$ a constant

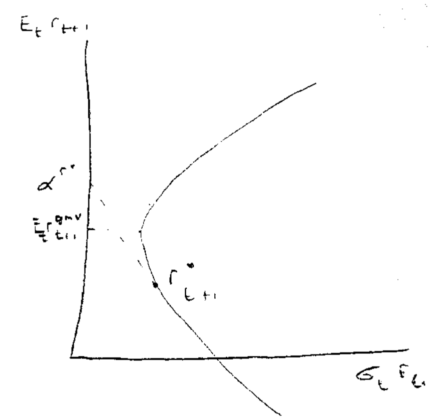
which is not a valid s.d.f $\Rightarrow$ r^f_{t+1} not a valid Beta pricing model

• if $r^f_{t+1} \notin R_{t+1}$ then

1) each r^{mv}_{t+1} gives the m^*_{t+1} that corresponds to an augmented space

$R^a_{t+1} = R_{t+1} + \{\alpha_{mv}\}$





2) $\alpha^{r^*} = \frac{1}{E_t m^*_{t+1}}$

$= \frac{E_t(r^{*2}_{t+1})}{E_t(r^*_{t+1})} \neq E_t[r^{gmv}_{t+1}] = \frac{E_t[r^*_{t+1}]}{1 - E_t[z^*_{t+1}]}$ cf. FNI p 49

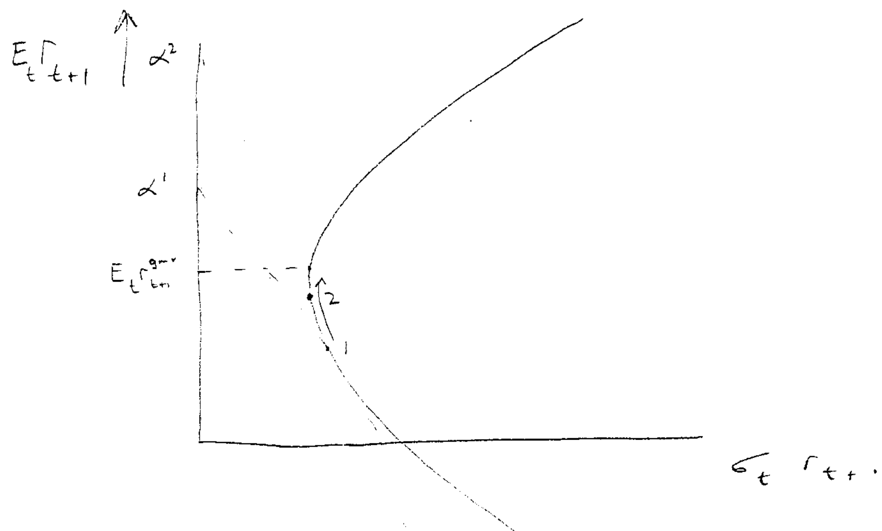
5

3) $w = \dfrac{E_t[r^{*2}_{t+1}]}{E_t[r^{*}_{t+1}]}$ corresponds to $E_t m_{t+1} = \infty$ & so m_{t+1} not a valid sdf

& $\alpha_{mv} = 0$

$w = \dfrac{E_t[r^{*}_{t+1}]}{1 - E_t[z^{*}_{t+1}]} = w^{gmv}$ corresponds to $E_t m_{t+1} = 0$

& $\alpha_{mv} = \infty$ or $-\infty$ so r^{mv}_{t+1} not a valid Beta-pricing model



- obtaining m^*_{t+1} from m_{t+1}
  - suppose $X_{t+1} = \{ \underline{w}'_t \underline{x}_{t+1} \mid \underline{w}_t \in \mathcal{F}_t \}$

    so $\underset{N\times 1}{\underline{x}_{t+1}}$ forms a basis for X_{t+1} w.r.t $\mathcal{F}_t$
  - know $m_{t+1} = m^*_{t+1} + e_{t+1}$

    where $E_t[e_{t+1} \underline{x}'_{t+1}] = \underline{0}'$
  - can write $m^*_{t+1} = \underline{c}' \underline{x}_{t+1}$
  - so

$$E_t[m_{t+1} \underline{x}'_{t+1}] = E_t[\underline{c}' \underline{x}_{t+1} \underline{x}'_{t+1}] + \underbrace{E_t[e_{t+1} \underline{x}']}_{0}$$

$$\Rightarrow \quad \underline{c}' = \underset{1\times N}{E_t[m_{t+1} \underline{x}'_{t+1}]} \, \underset{N\times N}{(E_t[\underline{x}_{t+1} \underline{x}'_{t+1}])^{-1}}$$

Beta Pricing Models

- let $E_{\hat{t}}[\,\cdot\,] \equiv E[\,\cdot\,|\hat{\mathcal{F}}_t]$

- for an arbitrary payoff space $\tilde{X}_{t+1}$ define the return space

$$\tilde{R}_{t+1} \equiv \{ r_{t+1} \in \tilde{X}_{t+1} \mid p_t(r_{t+1}) = 1 \}$$

- defn of a Beta pricing model, β-p-m

Defn: A set of factors $\underline{h}_{t+1}$ constitute a β-p-m wrt $\hat{\mathcal{F}}_t$ for $\tilde{R}_{t+1}$ iff

$$E_{\hat{t}}[r^i_{t+1}] = \hat{\alpha} + \hat{\underline{\lambda}}' \hat{\underline{\beta}}_{i,h} \quad \forall\ r^i_{t+1} \in \tilde{R}_{t+1}$$

where $\begin{bmatrix} \hat{\alpha} \\ \hat{\underline{\lambda}} \end{bmatrix} \in \hat{\mathcal{F}}_t$, $\hat{\underline{\beta}}_{i,h} = \left(E_{\hat{t}}[\tilde{\underline{h}}_{t+1}\tilde{\underline{h}}'_{t+1}]\right)^{-1} E_{\hat{t}}[\tilde{\underline{h}}_{t+1} r^i_{t+1}]$

& $\tilde{\underline{h}}_{t+1} = \underline{h}_{t+1} - E_{\hat{t}}[\underline{h}_{t+1}]$

- equivilence of a β-p-m & a linear sdf

Theorem: $\underline{h}_{t+1}$ constitutes a β-p-m wrt $\hat{\mathcal{F}}_t$ for $\tilde{R}_{t+1}$ $\iff$ $\exists$ $\hat{a}$ $\underline{b}' \in \hat{\mathcal{F}}_t$ $\ni$ $m_{t+1} = a + \underline{b}'\underline{h}_{t+1}$ is a valid sdf wrt $\hat{\mathcal{F}}_t$ for $\tilde{R}_{t+1}$ & so for

$$\hat{\tilde{X}}_{t+1} = \left\{ (\underline{\omega}^i_t)' \underline{r}^i_{t+1} \;\middle|\; \{\underline{r}^i_{t+1}\} \subset \tilde{R}_{t+1}\ \ \underline{\omega}^i_t \in \hat{\mathcal{F}}_t \right\}$$

(so letting $\hat{\underline{r}}_{t+1} \in \tilde{R}_{t+1}$ span $\tilde{R}_{t+1}$ wrt $\hat{\mathcal{F}}_t$,

$$\hat{X}_{t+1} = \{ \underline{\omega}'_t \hat{\underline{r}}_{t+1} \mid \underline{\omega}_t \in \hat{\mathcal{F}}_t \})$$

1

$\Rightarrow$ let $E_{\hat{t}}[m_{t+1}] = \hat{a} + \underline{b}' E_{\hat{t}}[h_{t+1}] = a$

& so $m_{t+1} = \hat{a} + \underline{b}' \underline{h}_{t+1} = a + \underline{b}' \tilde{h}_{t+1}$

pf:

$m_{t+1} = a + \underline{b}' \tilde{h}_{t+1}$ an sdf wrt $\hat{\mathcal{F}}_t$ & $\hat{\tilde{X}}_{t+1}$ as defined in the theorem

$$\iff E_{\hat{t}}[m_{t+1} x^i_{t+1}] = E_{\hat{t}}[p_t(x^i_{t+1})] \quad \forall x^i_{t+1} \in \tilde{X}_{t+1}$$

$$\iff E_{\hat{t}}[m_{t+1} (\underline{w}^i_{t+1})' \underline{r}^i_{t+1}] = (\underline{w}^i_{t+1})' \underline{i} \quad \forall \underline{w}^i_{t+1} \in \hat{\mathcal{F}}_t, \ \forall \{\underline{r}^i_{t+1}\} \subset \tilde{R}_{t+1}$$

$$\iff E_{\hat{t}}[m_{t+1} r^i_{t+1}] = 1 \quad \forall r^i_{t+1} \in \tilde{R}_{t+1}$$

$$\iff 1 = E_{\hat{t}}[m_{t+1}] E_{\hat{t}}[r^i_{t+1}] + \text{cov}_{\hat{t}}[m_{t+1}, r^i_{t+1}] \quad \forall r^i_{t+1} \in \tilde{R}_{t+1}$$

$$\iff E_{\hat{t}}[r^i_{t+1}] = \frac{1}{E_{\hat{t}}[m_{t+1}]} - \frac{E_{\hat{t}}[\underline{b}' \underline{h}_{t+1} r^i_{t+1}]}{E_{\hat{t}}[m_{t+1}]}$$

$$= \underbrace{\frac{1}{a}}_{\hat{\alpha}} + \underbrace{-\frac{1}{a} \underline{b}' E_{\hat{t}}[\tilde{\underline{h}}_{t+1} \tilde{\underline{h}}'_{t+1}]}_{\hat{\underline{\lambda}}'} \underbrace{\left(E_{\hat{t}}[\tilde{\underline{h}}_{t+1} \tilde{\underline{h}}'_{t+1}]\right)^{-1} \left(E_{\hat{t}}[\tilde{\underline{h}}_{t+1} r^i_{t+1}]\right)}_{\hat{\beta}_{i,h}}$$

$$\forall r^i_{t+1} \in \tilde{R}_{t+1}$$

- demeaning the factors is unimportant
- note that since $\hat{\tilde{X}}_{t+1}$ can be a subset of $\tilde{X}_{t+1}$, $\underline{h}_{t+1}$ constitutes a β-pm wrt $\hat{\mathcal{F}}_t$ for $\tilde{R}_{t+1} \not\Rightarrow \exists\ a, \underline{b}' \in \hat{\mathcal{F}}_t \ni m_t = a + \underline{b}' \underline{h}_{t+1}$ is a valid sdf wrt $\hat{\mathcal{F}}_t$ for $\tilde{X}_{t+1}$

(2)

- multi-factor β-p-m implies a single-factor β-p-m

. from above, $\underline{h}_{t+1}$ a multi-factor β-p-m for $\tilde{R}_{t+1}$ $\Rightarrow$ $m_{t+1} = a + \underline{b}'\tilde{\underline{h}}_{t+1}$ a valid sdf for $\tilde{R}_{t+1}$

. take $g_{t+1} = \underline{b}'\underline{h}_{t+1}$ (so $\tilde{g}_{t+1} = \underline{b}'\tilde{\underline{h}}_{t+1}$)

. know $m_{t+1} = a + \tilde{g}_{t+1}$ is a valid sdf for $\tilde{R}_{t+1}$

so from above know g_{t+1} constitutes a single-factor β-p-m for $\tilde{R}_{t+1}$

③

- mimicking portfolios

· suppose $\underline{h}_{t+1}$ constitutes a multifactor
$\quad\ _{K\times 1}$

β-p-m for $\tilde{R}_{t+1}$ wrt $\hat{\mathcal{G}}_t$

· again let $\underset{n\times 1}{\underline{\hat{r}}_{t+1}} \in \tilde{R}_{t+1}$ span $\tilde{R}_{t+1}$ wrt $\hat{\mathcal{G}}_t$ & again

define $\hat{\tilde{X}}_{t+1} = \left\{ \underline{w}_t' \ \underline{\hat{r}}_{t+1} \mid \underline{w}_t \in \hat{\mathcal{G}}_t \right\}$

- can obtain K assets, each an element of $\hat{\tilde{X}}_{t+1}$ that also constitute a mult-factor Beta pricing model for $\tilde{R}_{t+1}$ w.r.t $\hat{\mathcal{F}}_t$
- these assets are known as mimicking portfolios
- from above, know $a + \underline{b}'\underline{\tilde{h}}_{t+1}$ is a valid s.d.f for $\tilde{R}_{t+1}$
- project each element of $\underset{K\times 1}{\underline{\tilde{h}}_{t+1}}$ on $\underset{M\times 1}{\hat{\underline{r}}_{t+1}}$ w.r.t. $\hat{\mathcal{F}}_t$
- let $\underset{K\times 1}{\underline{x}^h_{t+1}}$ be the set of assets obtained

from the projection: so $\underset{K\times 1}{\underline{x}^h_{t+1}} = \underset{K\times M}{Q}\ \underset{M\times 1}{\hat{\underline{r}}_{t+1}}$

where $Q = \underset{K\times M}{E_{\hat{t}}[\underline{\tilde{h}}_{t+1}\hat{\underline{r}}'_{t+1}]}\left(\underset{M\times M}{E_{\hat{t}}[\hat{\underline{r}}_{t+1}\hat{\underline{r}}'_{t+1}]}\right)^{-1} \in \hat{\mathcal{F}}_t$

& $x^h_{t+1} \in \hat{\tilde{X}}_{t+1}$

- know $\underline{\tilde{h}}_{t+1} = \underline{x}^h_{t+1} + \underline{e}^h_{t+1}$ where $E_{\hat{t}}[\underline{e}^h_{t+1}\hat{\underline{r}}'_{t+1}] = \underline{0}$
- so $E_{\hat{t}}\left[(a + \underline{b}'\underline{\tilde{h}}_{t+1})\, r^i_{t+1}\right] = E_{\hat{t}}\left[(a + \underline{b}'\underline{x}^h_{t+1})\, r^i_{t+1}\right] + \underbrace{E_{\hat{t}}[\underline{b}'\underline{e}^h_{t+1}\hat{\underline{r}}'_{t+1}]}_{\underline{0}}\underline{w}^i$

$= 1 \quad \forall\, r^i_{t+1} \in \tilde{R}_{t+1}$

5

- can normalize the elements of $\underline{x}^h_{t+1}$ to be returns $\underline{r}^h_{t+1}$ & then can write

$$E_t[(a + \hat{\underline{b}}' \underline{r}^h_{t+1}) r^i_{t+1}] = 1 \qquad \forall\ r^i \in \tilde{R}_{t+1}$$

- from above $\underline{r}^h_{t+1}$ constitutes a beta pricing model for $\tilde{R}_{t+1}$ wrt $\hat{\mathcal{F}}_t$ (recalling that $a + \hat{\underline{b}}' \underline{r}^h_{t+1}$ can be rewritten $\hat{a} + \hat{\underline{b}}' \tilde{\underline{r}}^h_{t+1}$ where $\tilde{r}^h_{t+1} = \underline{r}^h_{t+1} - E_{\hat{t}}[\underline{r}^h_{t+1}]$)

- note that can define $\tilde{R}_{t+1}$ to be

$$R_{t+1} = \{ x_{t+1} \in X_{t+1} \mid p_t(x_{t+1}) = 1 \}$$

where $\underline{r}_{t+1} \in R_{t+1}$ spans R_{t+1} wrt $\mathcal{F}_t$ & can set $\hat{\mathcal{F}}_t$ to be $\mathcal{F}_t$