Conditional vs Unconditional Beta Pricing Models

0

- define X_{t+1} & R_{t+1} as before; assume $\underline{r}_{t+1} \in R_{t+1}$ spans X_{t+1} wrt $\mathcal{F}_t$; so

$$X_{t+1} = \left\{ \underline{w}_t' \underline{r}_{t+1} \;\middle|\; \underline{w}_t \in \mathcal{F}_t \right\}$$

$$R_{t+1} = \left\{ \underline{w}_t' \underline{r}_{t+1} \;\middle|\; \underline{w}_t \in \mathcal{F}_t \quad \underline{w}_t' \underline{i} = 1 \right\}$$

- define X^0_{t+1} & R^0_{t+1} as follows

$$X^0_{t+1} = \left\{ \underline{w}' \underline{r}_{t+1} \;\middle|\; \underline{w} \in \mathcal{F}^0_t \right\}$$

$$R^0_{t+1} = \left\{ \underline{w}' \underline{r}_{t+1} \;\middle|\; \underline{w} \in \mathcal{F}^0_t \quad \underline{w}' \underline{i} = 1 \right\}$$

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- using unconditional moments
  - if m_{t+1} is a sdf for X_{t+1} wrt $\mathcal{F}_t$

then

$$E[\,E_t[m_{t+1}x_{t+1} \mid \mathcal{F}_t]\,] = E[\,p_t(x_{t+1})\,]$$

$$\forall\ x_{t+1} \in X_{t+1}$$

$$E[\,m_{t+1}x_{t+1}\,] = E[\,p_t(x_{t+1})\,]$$

$$\forall\ x_{t+1} \in X_{t+1}$$

so m_{t+1} is a sdf for X_{t+1} wrt $\mathcal{F}^o_t$

- what about the other direction? yes

Theorem:

m_{t+1} is a sdf for X_{t+1} using $\mathcal{F}^o_t$

$\Rightarrow$ m_{t+1} is a sdf for X_{t+1} using $\mathcal{F}_t$

Pf: m_{t+1} is a sdf for X_{t+1} using $\mathcal{F}^o_t$ iff

$$E[\,(m_{t+1}x_{t+1} - p_t(x_{t+1}))\,] = 0 \quad \forall\ x_{t+1} \in X_{t+1}$$

$$\Rightarrow E[\,\underline{z}_t'(m_{t+1}\underline{r}_{t+1} - \underline{\iota})\,] = 0 \quad \forall\ \underline{z}_t \in \mathcal{F}_t$$

$$\Rightarrow E_t[\,m_{t+1}\underline{r}_{t+1} - \underline{\iota}\,] = 0$$

$$\Rightarrow E_t[\,m_{t+1}\underline{r}_{t+1}\,] = \underline{\iota}$$

$$\Rightarrow E_t[\,m_{t+1}x_{t+1}\,] = p_t(x_{t+1})$$

$$\forall\ x_{t+1} \in X_{t+1}$$

- conditional vs unconditional models

• Defn of a conditional Beta pricing model. A set of K factors $\underline{h}_{t+1}$ constitute a conditional Beta pricing model for $\tilde{R}_{t+1}$ if

$$E_t\, r^i_{t+1} = \alpha + \underline{\lambda}' \beta_{i,h} \quad \forall\, r^i_{t+1} \in \tilde{R}_{t+1}$$

where $\begin{bmatrix}\alpha \\ \underline{\lambda}\end{bmatrix} \in \mathcal{G}_t$ & $\beta_{i,h} = \left(\mathrm{cov}_t[h_{t+1}, h'_{t+1}]\right)^{-1} \mathrm{cov}_t[h_{t+1}, r^i_{t+1}]$

• Defn of an unconditional Beta pricing model. A set of K factors $\underline{h}_{t+1}$ constitute an uncond. β-p-m for $\tilde{R}_{t+1}$ if

$$E\, r^i_{t+1} = \alpha^o + \underline{\lambda}^{o\prime} \beta^o_{i,h} \quad \forall\, r^i_{t+1} \in \tilde{R}_{t+1}$$

where $\begin{bmatrix}\alpha^o \\ \underline{\lambda}^o\end{bmatrix} \in \mathcal{G}^o_t$ (ie $\begin{bmatrix}\alpha^o \\ \underline{\lambda}^o\end{bmatrix} \in \mathbb{R}^{K+1}$) & $\beta^o_{i,h} = \left(\mathrm{cov}[h_{t+1}, h'_{t+1}]\right)^{-1} \mathrm{cov}[h_{t+1}, r^i_{t+1}]$

• Theorem: $\underline{h}_{t+1}$ constitutes an unconditional β-p-m for R_{t+1}
$\Longrightarrow$ $\underline{h}_{t+1}$ constitutes a conditional β-p-m for R_{t+1}

Pf. $\underline{h}$ constitutes an uncond β-p-m for R_{t+1}

$\Longleftrightarrow$ $\exists\ \hat{a}$ & $\underline{b} \in \mathcal{G}^o_t \ni E[(\hat{a} + \underline{b}' h_{t+1})\, r_{t+1}] = 1$

$\forall\, r_{t+1} \in R_{t+1}$

$\Longleftrightarrow$ $E[((\hat{a} + \underline{b}' \underline{h}_{t+1})\, \underline{r}_{t+1} - \underline{\iota})\, z_t] = 0 \quad \forall\, z_t \in \mathcal{G}_t$

$\Longleftrightarrow$ $E_t[(\hat{a} + \underline{b}' \underline{h}_{t+1})\, \underline{r}_{t+1}] = 1$

$\Longleftrightarrow$ $E_t[\hat{a} + \underline{b}' \underline{h}_{t+1})\, r_{t+1}] = 1 \quad \forall\, r_{t+1} \in R_{t+1}$

$\Longrightarrow$ $\underline{h}_{t+1}$ constitutes a cond β-p-m for R_{t+1}

• note
1. important that h_{t+1} constitutes an uncond β-p-m for R_{t+1} & not just R^o_{t+1} for Theorem to hold
(3) ($\neq$) can be shown for R^o_{t+1}: see counter eg B below)

• Theorem $\underline{h}_{t+1}$ constitutes an uncond β-p-m for R_{t+1}

$\not\Leftarrow$ $\underline{h}_{t+1}$ constitutes a cond β-p-m for R_{t+1}

Pf: $\underline{h}_{t+1}$ constitutes a cond β-p-m for X_{t+1}

$\Rightarrow$ $\exists$ $\hat{a}$ & $\underline{b} \in \mathcal{G}_t$ $\ni$

$$E_t[(\hat{a} + \underline{b}'\underline{h}_{t+1})\, r_{t+1}] = 1 \quad \forall r_{t+1} \in R_{t+1}$$

$$\Rightarrow E[(\hat{a} + \underline{b}' h_{t+1})\, r_{t+1}] = 1 \quad \forall r_{t+1} \in R_{t+1}$$

$$\not\Rightarrow \exists\, \hat{a}^0 \,\&\, \underline{b}^0 \in \mathcal{G}_t^0 \ni E[(\hat{\underline{a}}^0 + \underline{b}^{0\prime}\underline{h}_{t+1})\, r_{t+1}] = E[p_t(x_{t+1})] \quad \forall r_{t+1} \in R_{t+1}$$

But need a counterexample to complete the proof see counterex A below

• Theorem $\underline{h}_{t+1}$ constitutes a cond β-p-m for R_{t+1}

$\ni$ $E_t r^i_{t+1} = \alpha + \underline{\lambda}'\beta_{ih}$ $\forall r^i_{t+1} \in R_{t+1}$ &

$$\hat{a} = \frac{1}{\alpha}\left[1 + \underline{\lambda}'(\mathrm{cov}_t[\underline{h}_{t+1}, \underline{h}'_{t+1}])^{-1} E_t[\underline{h}_{t+1}]\right] \in \mathcal{G}^0_t \quad \& \quad (1)$$

$$\underline{b}' = -\underline{\lambda}'(\mathrm{cov}_t[\underline{h}_{t+1}, \underline{h}'_{t+1}])^{-1}\frac{1}{\alpha} \in \mathcal{G}^0_t$$

$\Rightarrow$ $\underline{h}_{t+1}$ constitutes a uncond β-p-m for R_{t+1}

Pf $E_t r^i_{t+1} = \alpha + \underline{\lambda}'\beta_{ih}$ $\forall r^i_{t+1} \in R_{t+1}$

$$\Leftrightarrow E_t[(\hat{a} + \underline{b}' h_{t+1})\, x_{t+1}] = p_t(x_{t+1}) \quad \forall x_{t+1} \in X_{t+1}$$

$$\Leftrightarrow E[(\hat{a} + \underline{b}' h_{t+1})\, x_{t+1}] = E[p_t(x_{t+1})] \quad \forall x_{t+1} \in X_{t+1} \quad (2)$$

(1) & (2) $\Leftrightarrow$ $\underline{h}_{t+1}$ constitutes an uncond β-p-m for R_{t+1}

• note

1. (1) can hold when α, $\underline{\lambda}$ & $\beta_{ih} \notin \mathcal{G}^0_t$

2. (1) holds when $E_t[\cdot]$ & $E[\cdot]$ are the same

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- conditional vs unconditional min-var frontier

Theorem: r^{mv}_{t+1} on the uncond min-var frontier for R_{t+1}

$\overset{\not\Leftarrow}{\Rightarrow}$ r^{mv}_{t+1} on the cond min-var frontier for R_{t+1}

Pf: r^{mv}_{t+1} on uncond min-var frontier for R_{t+1}

$\Leftrightarrow$ $E[(a^\circ + b^\circ r^{mv}_{t+1}) r_{t+1}] = 1$ $\quad \forall r_{t+1} \in R_{t+1} \quad a^\circ, b^\circ \in \mathcal{F}^\circ_t$

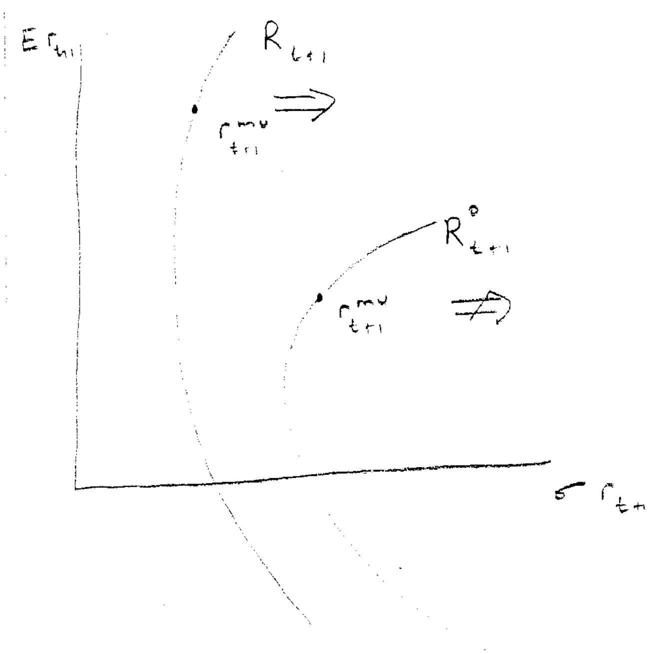
$\Leftrightarrow$ $E_t[(a^\circ + b^\circ r^{mv}_{t+1}) r_{t+1}] = 1$ $\quad \forall r_{t+1} \in R_{t+1} \quad a^\circ, b^\circ \in \mathcal{F}^\circ_t$

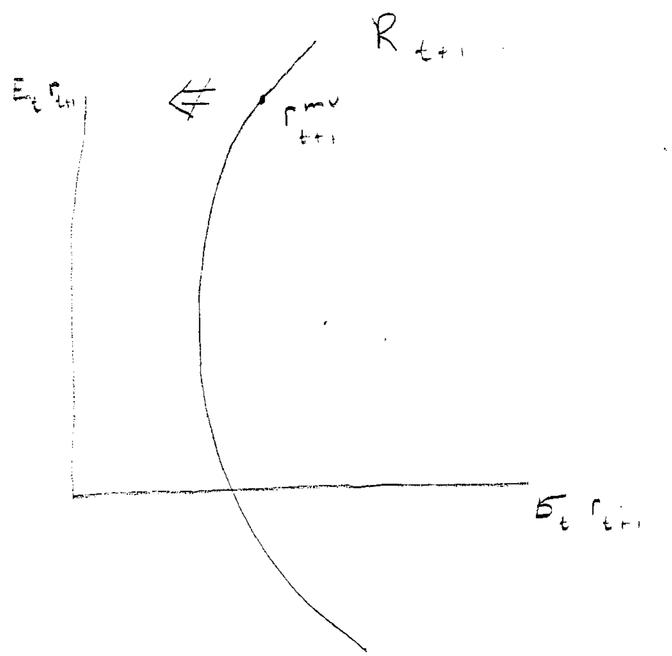
$\overset{\not\Leftarrow}{\Rightarrow}$ r^{mv}_{t+1} on cond min-var fontier for X_{t+1}

· note that

r^{mv}_{t+1} on the uncond min-var frontier for R°_{t+1}

$\overset{\not\Leftarrow}{\not\Rightarrow}$ r^{mv}_{t+1} on the cond min-var frontier for R_{t+1}





· note

r^{mv}_{t+1} on the uncond min-var frontier for R°_{t+1}

$\not\Leftarrow$

$\not\Rightarrow$ r^{mv}_{t+1} on the cond min-var frontier for R°_{t+1}

9 (for $\Rightarrow$ see counter eg B below & for $\Leftarrow$ see counter eg A)

· brute force proof of $\Rightarrow$ part of Theorem which can be restated

r_{t+1} not on the cond MVF for R_{t+1}

$\Rightarrow$ r_{t+1} not on the uncond MVF for R_{t+1}

Suppose wlg $\mathcal{F}_t$ is generated by $z_t \in Z_t$

& that $\hat{r}_{t+1}$ is not on cond MVF for R_{t+1}

So letting $\hat{Z}_t$ be the set of $z_t \ni \hat{r}_{t+1}$ is not on the MVF, $\text{Prob}\{z_t \in \hat{Z}_t\} > 0$ using $\mathcal{F}_t^\circ$

Now $\exists\ \hat{\omega}(z_t) \ni$ defining $\hat{\hat{r}}_{t+1} = \hat{\omega}(z_t)\, \underline{r}_{t+1}$

$$E_t\, \hat{\hat{r}}_{t+1} = E_t\, \hat{r}_{t+1} \quad \forall\ z_t \in Z_t \qquad (1)$$

$$\left.\begin{array}{ll} \sigma_t\, \hat{\hat{r}}_{t+1} < \sigma_t\, \hat{r}_{t+1} & \forall\ z_t \in \hat{Z}_t \\ \sigma_t\, \hat{\hat{r}}_{t+1} \leq \sigma_t\, \hat{r}_{t+1} & \forall\ z_t \notin \hat{Z}_t \end{array}\right\} \quad (2)$$

$$(1) \Rightarrow \quad E\, \hat{\hat{r}}_{t+1} = E\, \hat{r}_{t+1} \qquad (3)$$

$$(1)\ \&\ (2) \Rightarrow \left\{\begin{array}{ll} E_t\, \hat{\hat{r}}^2_{t+1} < E_t\, \hat{r}^2_{t+1} & \forall\ z_t \in \hat{Z}_t \\ E_t\, \hat{\hat{r}}^2_{t+1} \leq E_t\, \hat{r}^2_{t+1} & \forall\ z_t \notin \hat{Z}_t \end{array}\right\} \Rightarrow E\, \hat{\hat{r}}^2_{t+1} < E\, \hat{r}^2_{t+1} \qquad (4)$$

since $\text{Prob}\{z_t \in \hat{Z}_t\} > 0$ using $\mathcal{F}_t^\circ$

$$(3)\ \&\ (4) \Rightarrow \quad \sigma\, \hat{\hat{r}}_{t+1} < \sigma\, \hat{r}_{t+1} \qquad (5)$$

& so $\hat{r}_{t+1}$ not on uncond MVF for R_{t+1}

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· why does $\Rightarrow$ proof break down with R°_{t+1}

Note that $\hat{r}_{t+1}$ formed using weight vector $\hat{w}(z_t) \in \mathcal{F}_t$. Unless $\hat{w}(z_t) = \hat{w}$, a constant vector $\forall z_t \in Z_t$ $\hat{r}_{t+1}$ does not belong in R°_{t+1}

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· using the decomposition to prove ⇒

1 $E_t[x^*_{t+1}\ x_{t+1}] = p_t(x_{t+1}) \quad \forall\, x_{t+1} \in X_{t+1}$

$\Rightarrow E[x^*_{t+1}\ x_{t+1}] = E[p_t(x_{t+1})] \quad \forall\, x_{t+1} \in X_{t+1}$

2 $E_t[r^*_{t+1}\ r_{t+1}] = E_t[r^{*2}_{t+1}] \quad \forall\, r_{t+1} \in R_{t+1}$

$\Rightarrow E[r^*_{t+1}\ r_{t+1}] = E[r^{*2}_{t+1}] \quad \forall\, r_{t+1} \in R_{t+1}$

3 $E_t[z^*_{t+1}\ z_{t+1}] = E_t[z_{t+1}] \quad \forall\, z_{t+1} \in Z_{t+1}$

$\Rightarrow E[z^*_{t+1}\ z_{t+1}] = E[z_{t+1}] \quad \forall\, z_{t+1} \in Z_{t+1}$

4 can show that any return on the MVF wrt $\mathcal{F}^0_t$ can be written

$$r^{mv}_{t+1} = r^*_{t+1} + w\, x^*_{t+1} \qquad w \in \mathcal{F}^0_t$$

5 any return on the MVF wrt $\mathcal{F}_t$ can be written

$$r^{mv}_{t+1} = r^*_{t+1} + w_t \cdot x^*_{t+1} \qquad w_t \in \mathcal{F}_t$$

8

· counter eg A to illustrate ⇍ part of the Theorem

suppose $r^f_{t+1} \in \mathcal{F}^o_t$

two possible states given $\mathcal{F}_t$, each equally likely

r^p_{t+1} lies on the conditional MVF

$E[r^p_{t+1} | s_1] = r^f_{t+1} + \delta \qquad \sigma^2[r^p_{t+1} | s_1] > 0$

$E[r^p_{t+1} | s_2] = r^f_{t+1} - \delta \qquad \sigma^2[r^p_{t+1} | s_2] > 0$

$\Rightarrow \quad E[r^p_{t+1}] = r^f_{t+1}$ but

$$\sigma^2[r^p_{t+1}] = \underbrace{E[\sigma^2_t[r^p_{t+1}]]}_{>0} + \underbrace{\sigma^2[E_t[r^p_{t+1}]]}_{\geq 0}$$

> 0

& so r^p_{t+1} not on uncond. MVF

(this counter eg also works for R^o_{t+1})

⑨

• Counterexample B to illustrate $\not\Rightarrow$ when R^o_{t+1} & not R_{t+1} $\quad Pr\{1\} = Pr\{2\} = 0.5$

$r_{f11} = 1 = r_{f12} \quad \mu_{A11} = 1.1 = \mu_{A12} \quad \sigma^2_{A11} = 0.01 = \sigma^2_{A12}$

$\mu_{B11} = 1.3 \quad \mu_{B12} = 0.7 \quad \sigma^2_{B11} = 0.01 = \sigma^2_{B12}$

$\sigma_{AB11} = \sigma_{AB12} = 0$

$$w_{P11} = \begin{bmatrix} \frac{0.1}{0.01} \\ \frac{0.3}{0.01} \end{bmatrix} \frac{1}{10+30} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}$$

$$w_{P12} = \begin{bmatrix} \frac{0.1}{0.01} \\ \frac{-0.3}{0.01} \end{bmatrix} \frac{1}{10-30} = \begin{bmatrix} -0.5 \\ 1.5 \end{bmatrix}$$

With a riskfree asset, & 2 assets A & B uncorrelated the tangency portfolio P given by

$$w_p = \frac{V^{-1}(\underline{\mu} - r_f \underline{i})}{\underline{i}' V^{-1}(\underline{\mu} - r_f \underline{i})}$$

$$= \begin{bmatrix} \frac{\mu_A - r_f}{\sigma_A^2} \\ \frac{\mu_B - r_f}{\sigma_B^2} \end{bmatrix} \frac{1}{\left(\frac{\mu_A - r_f}{\sigma_A^2} + \frac{\mu_B - r_f}{\sigma_B^2}\right)}$$

since $V = \begin{bmatrix} \sigma_A^2 & 0 \\ 0 & \sigma_B^2 \end{bmatrix}$

$V^{-1} = \begin{bmatrix} \frac{1}{\sigma_A^2} & 0 \\ 0 & \frac{1}{\sigma_B^2} \end{bmatrix}$

$\mu_A = 1.1 \quad \sigma_A^2 = 0.01$

$\mu_B = 1 \quad \sigma_B^2 = E[\sigma^2_{B1s}] + \sigma^2[\mu_{B1s}]$

$$= 0.01 + \left(\tfrac{1}{2}\, 0.3^2 + \tfrac{1}{2}\, 0.3^2\right)$$

$$= 0.01 + 0.09 = 0.1$$

$\sigma_{AB} = E[\sigma_{AB1s}] + \sigma[\mu_{A1s}, \mu_{B1s}] = 0 + 0 = 0$

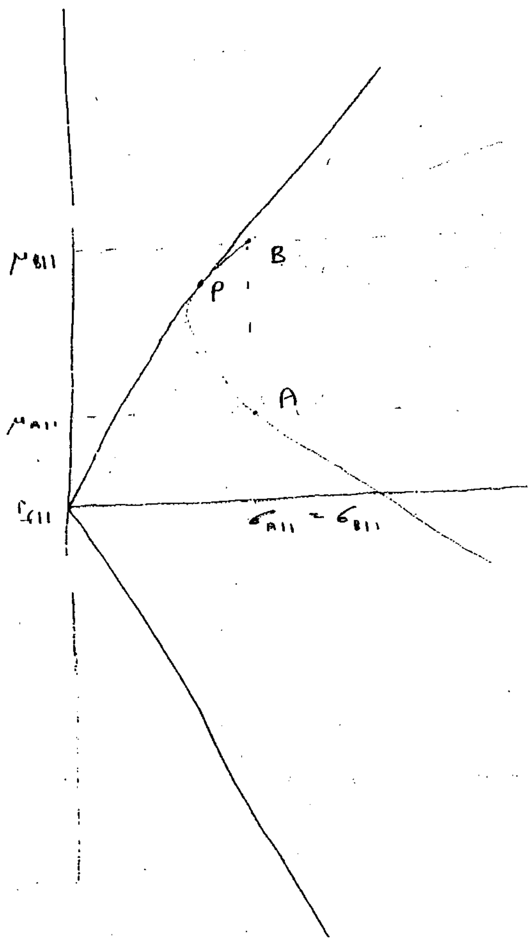
$$w_p = \begin{bmatrix} \frac{0.1}{0.01} \\ \frac{0}{0.1} \end{bmatrix} \frac{1}{10} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$P = A$

A on uncond MVF

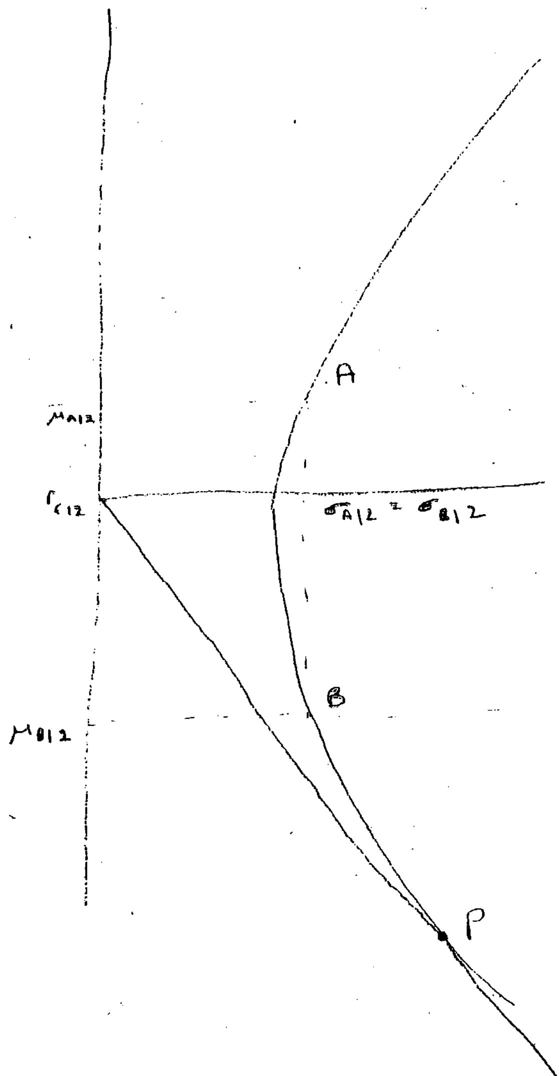
but <u>not</u> on cond MVF

(10)



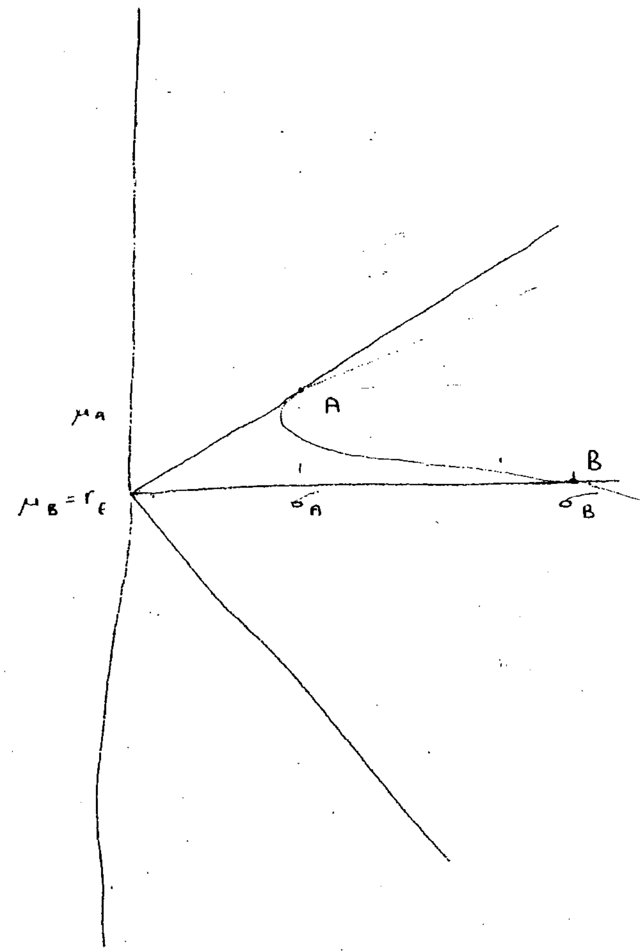
μ_{B11}
μ_{A11}
r_{f11}
P
B
A
$\sigma_{A11} = \sigma_{B11}$

S1



μ_{A12}
r_{f12}
μ_{B12}
A
B
P
$\sigma_{A12} = \sigma_{B12}$

S2



μ_A
$\mu_B = r_f$
A
B
σ_A
σ_B

Uncond

Utility Theory

- utility theory single period
  - wish to use a concept of utility that is able to deal with uncertainty
  - introduce von Neumann - Morgenstern utility function
    1. the investor makes choices consistent with maximizing the expected value of the utility function
    2. utility is defined over either consumption or wealth but can also be defined over a bundle of goods
    3. von Neumann - Morgenstern utility fcns can be developed axiomatically

suppose an individual cares about consumption c & define a lottery L as a set of consumption outcomes $(c_1, \quad, c_m)$ & an associated set of probabilities $(\pi_1, \quad, \pi_m)$. Let $\mathcal{L}$ be the space of lotteries. Elements of the set of consumption outcomes can be elements of $\mathcal{L}$

Axiom 1 (completeness) For every pair of lotteries $L_1 \succsim L_2$ or $L_1 \precsim L_2$

Axiom 2 (reflexivity) For every lottery, $L \succsim L$

Axiom 3 (transitivity) If $L_1 \succsim L_2$ & $L_2 \succsim L_3$, then $L_1 \succsim L_3$

Axiom 4 (continuity) For any $x, y, z \in \mathcal{L}$,

$\{ \pi \in [0,1] : ((x, y), (\pi, (1-\pi))) \succsim (z, 1) \}$ &

$\{ \pi \in [0,1] : ((x, y), (\pi, (1-\pi))) \precsim (z, 1) \}$ are closed sets

Axiom 5 (independence) If $(x,1) \sim (y,1)$ then $((x,z), (\pi, 1-\pi)) \sim ((y,z), (\pi, 1-\pi))$

Axiom 7 (<u>dont</u> need) If $x_1 \succsim x_2 \succsim x_3$ then $\exists\ \pi \in [0,1] \ni (x_2, 1) \sim ((x_1, x_3), (\pi, 1-\pi))$. The π is unique unless $x_1 \sim x_3$

Axiom 6 (dominance) Let $L_1 = ((x_1, x_2), (\pi_1, 1-\pi_1))$ & $L_2 = ((x_1, x_2), (\pi_2, 1-\pi_2))$ If $x_1 \succ x_2$ then $L_1 \succ L_2 \Leftrightarrow \pi_1 > \pi_2$

Theorem: Under Axioms 1-6, $\exists$ a utility fcn $u(\cdot) \ni$ the choice of lottery by a decision-maker corresponds to the lottery with the highest $E[U(\cdot)]$

Pf Let L_h be the best lottery in $\mathcal{L}$ & L_w be the worst lottery in $\mathcal{L}$

Define $u(L_h) = 1$ & $u(L_w) = 0$ To find the utility of an intermediate lottery $L_i \in \mathcal{L}$ set $u(L_i) = p_i$ where p_i defined by

$$((L_h, L_w), (p_i, 1-p_i)) \sim (L_i, 1)$$

Need to check 2 things

1 existence of such a p_i: The two sets

$$\{ p \in [0,1] \; ((L_h, L_w), (p, 1-p)) \succsim L_i \} \;\&$$

$$\{ p \in [0,1] \; ((L_h, L_w), (p, 1-p)) \precsim L_i \}$$

are closed (Axiom 4) & nonempty (since L_h best & L_w worst) & every point in $[0,1]$ is in one or the other of the two sets (Axiom 1) Since the unit interval is connected, there must be some p in both but this will be just the desired p_i

2 uniqueness of such a p_i: Suppose p_i & p_i' both satisfy the above defn Then one must be larger than the other By Axiom 6, the lottery that gives a bigger prob of L_h must be preferred to the one that gives a smaller prob. Hence p_i is unique & $u(\cdot)$ is well defined

We next check that $u(\cdot)$ has the expected utility property. This can be shown

$$((L_x, L_y), (p, 1-p)) \sim \Big(\big(((L_h, L_w)(p_x, 1-p_x)), ((L_h, L_w)(p_y, 1-p_y)) \big), (p, 1-p) \Big) \quad \text{Axiom 5}$$

$$\sim \Big((L_h, L_w) \big(p_x p + p_y(1-p), \overbrace{(1-p_x)p + (1-p_y)(1-p)}^{1-p_x p - p_y(1-p)} \big) \Big)$$

$$\sim \Big((L_h, L_w) \big(p\, u(L_x) + (1-p)\, u(L_y), 1 - p\, u(L_x) - (1-p)\, u(L_y) \big) \Big)$$

& so $u((L_x, L_y), (p, 1-p)) = p\, u(L_x) + (1-p) u(L_y)$

as required

Finally we verify that $u(\cdot)$ is a utility fcn

Suppose that $L_x \succ L_y$. Then

$u(L_x) = p_x \Rightarrow L_x \sim ((L_h, L_w), (p_x, 1-p_x))$

$u(L_y) = p_y \Rightarrow L_y \sim ((L_h, L_w), (p_y, 1-p_y))$

& so by Axiom 6 $u(L_x) > u(L_y)$

③

- a Von-Neumann - Morgenstern utility fcn is identified up to a pos linear transformation; ie $u(.)$ & $a + b\,u(.)$ with $b > 0$ are equivalent
- if prefer more to less, $u(.)$ is increasing
- risk aversion
  1. a decision-maker is said to be risk averse at a particular consumption level if she is not prepared to accept any actuarially fair, immediately resolved consumption gamble
  2. for a given VN-M utility fcn $u(.)$, the utility fcn is risk averse at c if $u(c) > E[u(c+\epsilon)]$ $\forall\ \epsilon \ni E[\epsilon] = 0$ & $var[\epsilon] > 0$

Theorem: a decision-maker is globally risk averse iff her VN-M utility fcn is strictly concave at all consumption levels

• absolute risk aversion

1. for a utility fcn $u(\cdot)$, the absolute risk aversion measure is defined

$$ARA(c) = -\frac{u''(c)}{u'(c)}$$

2. it measures the dollar amount that the decision-maker would pay to avoid an actuarially fair gamble (per unit of dollar gamble variance) ie

$$E[u(c+\epsilon)] \approx u(c - \underbrace{\tfrac{1}{2} ARA(c)\, var[\epsilon]}_{q})$$

where $E[\epsilon] = 0$

3. to show this, use Taylor series expansions of both sides

$$E\left[u(c) + \epsilon u'(c) + \tfrac{1}{2}\epsilon^2 u''(c)\right]$$

$$\approx u(c) - q\, u'(c)$$

$$\Rightarrow \tfrac{1}{2} var[\epsilon]\, u''(c) \approx -q\, u'(c)$$

$$q = \tfrac{1}{2} ARA(c)\, var[\epsilon]$$

(5) (6)

• relative risk aversion

1. for a utility fcn $u(\cdot)$, the relative risk aversion measure is defined

$$RRA(c) = -\frac{u''(c)}{u'(c)} c$$

2. it measures the amount (as a fraction of consumption) that the decision-maker would pay to avoid an actuarially fair gamble (per unit of variance of the gamble expressed as a fraction of consumption)

ie

$$E[u(c+\epsilon)] \approx u(c - \underbrace{c \tfrac{1}{2} RRA(c)\, var[\tfrac{\epsilon}{c}]}_{q})$$

where $E[\epsilon]=0$

3. to show this, notice that

$$\tfrac{1}{2} c\, RRA(c)\, var[\tfrac{\epsilon}{c}] = \tfrac{1}{2} ARA(c)\, var[\epsilon]$$

• comparison $ARA(c)$ + $RRA(c)$

1. consider constant $ARA(c)$ (so $RRA(c)$ is increasing in c)
   + constant $RRA(c)$ (so $ARA(c)$ is decreasing in c)

2. fix ϵ, vary c + consider the dollar payment the decision-maker would pay

$q_{CARA}(c) = \tfrac{1}{2}\, CARA\, var[\epsilon]$ a constant

$q_{CRRA}(c) = \tfrac{1}{2} \frac{CRRA}{c} var[\epsilon]$ which is decreasing in C

3. fix $\frac{\epsilon}{c}$ vary c + consider the payment as a fraction of c that the decision-maker would pay

$q_{CARA}(c)/c = \tfrac{1}{2} c\, CARA\, var[\tfrac{\epsilon}{c}]$ which is increasing in c

$q_{CRRA}(c)/c = \tfrac{1}{2}\, CRRA\, var[\tfrac{\epsilon}{c}]$ a constant

• HARA utility fcn

1 the functional form of the Hyperbolic Absolute Risk Aversion (HARA) utility is the following

$$u(w) = a\left(\frac{1-\varphi}{\varphi}\right)\left(\frac{w}{1-\varphi} - \hat{w}\right)^{\varphi} + b \quad \varphi \neq 0 \quad \frac{w}{1-\varphi} - \hat{w} > 0 \quad a > 0$$

with

$$u'(w) = a\left(\frac{w}{1-\varphi} - \hat{w}\right)^{\varphi-1} > 0$$

$$u''(w) = -a\left(\frac{w}{1-\varphi} - \hat{w}\right)^{\varphi-2} < 0$$

$$ARA(w) = -\frac{u''(w)}{u'(w)} = \left(\frac{w}{1-\varphi} - \hat{w}\right)^{-1}$$

$$RRA(w) = -\frac{u''(w)\,w}{u'(w)} = w\left(\frac{w}{1-\varphi} - \hat{w}\right)^{-1}$$

2. HARA specializes to a number of important utility specifications

3 if $\hat{w} = 0$ obtain power utility. defining $\gamma = 1-\varphi$

then

$$u(w) = \frac{w^{1-\gamma}}{1-\gamma} \quad \gamma > 0 \quad w > 0$$

& $RRA(w) = \gamma$

4 if $\hat{w} = 0$ & $\varphi \to 0$ obtain a special case of power utility with $RRA(w) = 1$

$$u(w) = \ln w \quad w > 0$$

5 if $\varphi = 2$ & $\hat{w} < 0$ obtain quadratic utility

$$u(w) = -(w^{*} - w)^2 \quad w < w^{*}$$

6 if $\varphi \to \infty$ & $\hat{w} < 0$, obtain exponential utility

7 $u(w) = -e^{-\lambda w}$ & $ARA(w) = \lambda$

3 if $\hat{w} = 0$ obtain power utility: defining $\gamma = 1-\varphi$
& setting $a = (1-\varphi)^{-1+\varphi}$, $1-\varphi > 0$ then

$$u(w) = \frac{w^{1-\gamma}}{1-\gamma} \quad \gamma > 0 \quad w > 0 \quad \& \quad RRA = \gamma$$

4 if $\hat{w} = 0$ $a = (1-\varphi)^{-1+\varphi}$ & $b = -\frac{1}{\varphi}$, letting $\varphi \to 0$
obtain log utility as the limit

$$U(w) = \frac{w^{\varphi} - 1}{\varphi}$$

so using L'Hopital's Rule

$$\lim_{\varphi \to 0} U(w) = \lim_{\varphi \to 0} \frac{\frac{d}{d\varphi}\left(e^{\varphi \ln w} - 1\right)}{\frac{d}{d\varphi}\varphi} = \lim_{\varphi \to 0} \frac{W^{\varphi} \ln W}{1} = \ln W$$

5 if $\varphi = 2$ & $\hat{w} < 0$ obtain quadratic utility:
setting $a = 1$ & $b = 0$ & defining $w^* = -\hat{w}$

$$U(W) = -\tfrac{1}{2}(W^* - W)^2 \qquad W < W^*$$

6 if $\hat{w} = \left(\frac{\varphi}{1-\varphi}\right)\frac{1}{\lambda}$ $a = \left(\frac{\varphi-1}{\varphi}\right)^{\varphi-1} \lambda^{\varphi}$ & $b = 0$

then letting $\varphi \to \infty$ obtain exponential utility

$$U(W) = \left(\frac{\varphi-1}{\varphi}\right)^{\varphi-1} \lambda^{\varphi} \frac{(1-\varphi)}{\varphi}\left(\frac{W}{1-\varphi} - \left(\frac{\varphi}{1-\varphi}\right)\frac{1}{\lambda}\right)^{\varphi}$$

$$= -\left(\frac{\varphi-1}{\varphi}\right)^{\varphi} \lambda^{\varphi}\left(\frac{1}{\varphi-1}\left(\frac{\varphi}{\lambda} - W\right)\right)^{\varphi}$$

$$= -\left(1 - \frac{\lambda W}{\varphi}\right)^{\varphi}$$

& so $\lim_{\varphi \to \infty} U(W) = \lim_{\varphi \to \infty} -\left(1 - \frac{\lambda W}{\varphi}\right)^{\varphi} = -e^{-\lambda W}$

& $ARA(W) = \lambda$

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- utility theory multiple periods
  - use an time separable (or additive) utility fcn

$$\max E\left[\sum_{\tau=0}^{\infty} \delta^{\tau} u(c_{t+\tau}) \mid \mathcal{F}_t\right]$$

where $u(.)$ is a vN-M utility fcn
δ is a patience parameter

- intertemporal elasticity of substitution for consumption

1. percentage change in consumption growth in response to a percentage change in one plus the interest rate

2. with power utility, the intertemporal elasticity of substitution for consumption is equal to the inverse of RRA to illustrate, ignore uncertainty + consider a 2-period problem

$$\frac{c_t^{1-\gamma}}{1-\gamma} + \delta \frac{c_{t+1}^{1-\gamma}}{1-\gamma} \quad \text{st} \quad c_{t+1} = (w_t - c_t)(1+R)$$

foc $\quad c_t^{-\gamma} - \delta(1+R)\, c_{t+1}^{-\gamma} = 0$

$$\Rightarrow \left(\frac{c_{t+1}}{c_t}\right)^{\gamma} = \delta(1+R)$$

$$\Rightarrow \frac{c_{t+1}}{c_t} = \delta^{\frac{1}{\gamma}} (1+R)^{1/\gamma}$$

$$\Rightarrow \ln \frac{c_{t+1}}{c_t} = \frac{1}{\gamma}\ln\delta + \frac{1}{\gamma}\ln(1+R)$$

Now the elasticity is given by

$$\frac{d \ln \frac{c_{t+1}}{c_t}}{d \ln(1+R)} = \frac{1}{\gamma} \quad \text{as required}$$

(9) (E)

Why?

$$\frac{d\ln\left(\frac{c_{t+1}}{c_t}\right)}{d\ln(1+R)} = \frac{\frac{d\ln\frac{c_{t+1}}{c_t}}{d\frac{c_{t+1}}{c_t}}}{\frac{d\ln(1+R)}{d(1+R)}} \frac{d\frac{c_{t+1}}{c_t}}{d(1+R)} = \frac{d\frac{c_{t+1}}{c_t} / \frac{c_{t+1}}{c_t}}{d(1+R)/(1+R)}$$

— using dynamic programming to solve multiperiod problems

- finite-lived investor

1. define

$$V_t(W_t, S_t) = \max_{\{c_\tau\}_{\tau=t}^{T-1} \{\underline{\alpha}_\tau\}_{\tau=t}^{T-1}} E\left[\sum_{\tau=t}^{T} \delta^{\tau-t} u(c_\tau) \,\middle|\, S_t, W_t\right]$$

$c_\tau, \underline{\alpha}_\tau \in \mathcal{F}(S_\tau, W_\tau)$

s.t. $W_{\tau+1} = (W_\tau - c_\tau)\left(\underline{\alpha}_\tau'(\underline{R}_{\tau+1} - \underline{i}_N R^f_{\tau+1}) + R^f_{\tau+1}\right) \quad \tau = t, \ldots, T-1$

where S_τ is the vector of state variables that affect the investor's happiness at time τ, $\tau = t, \ldots, T-1$

2. would like to convert multiperiod problem into a single period problem

3. solve recursively working back from T

$$V_{T-1}(W_{T-1}, S_{T-1}) = \max_{c_{T-1}\, \underline{\alpha}_{T-1}} E\left[u(c_{T-1}) + \delta\, u(W_T) \,\middle|\, S_{T-1}, W_{T-1}\right]$$

s.t. $W_T = (W_{T-1} - c_{T-1})\left(\underline{\alpha}_{T-1}'(\underline{R}_T - \underline{i}_N R^f_T) + R^f_T\right)$

$c_{T-1}, \underline{\alpha}_{T-1} \in \mathcal{F}(S_{T-1})$

$V_{T-2}(W_{T-2}, S_{T-2}) =$

$$\max_{\{c_\tau\}_{\tau=T-2}^{T-1} \{\underline{\alpha}\}_{\tau=T-2}^{T-1}} \left\{E\left[u(c_{T-2}) + \delta\, u(c_{T-1}) + \delta^2 u(W_T) \,\middle|\, S_{T-2}\right]\right\}$$

s.t. $W_{\tau+1} = (W_\tau - c_\tau)\left(\underline{\alpha}_\tau'(\underline{R}_{\tau+1} - \underline{i}_N R^f_{\tau+1}) + R^f_{\tau+1}\right) \quad \tau = T-2, T-1$

(10) $c_\tau, \underline{\alpha}_\tau \in \mathcal{F}(S_\tau)$

$$= \max_{c_{T-2}, \alpha_{T-2}} \left\{ E\left[u(c_{T-2}) + \delta \underbrace{\max_{c_{T-1}, \alpha_{T-1}} \left\{ \begin{array}{l} E\left[u(c_{T-1}) + \delta\, u(W_T) \mid s_{T-1}\right] \\ \text{st } W_T = (W_{T-1} - c_{T-1})\left(\alpha_{T-1}'(\underline{R}_T - \underline{i}_N R_T^f) + R_T^f\right) \\ c_{T-1}\ \alpha_{T-1} \in \mathcal{F}(s_{T-1}) \end{array} \right\}}_{V_{T-1}(W_{T-1},\, s_{T-1})} \Big| s_{T-2} \right] \right\}$$

st

$$W_{T-1} = (W_{T-2} - c_{T-2})\left(\alpha_{T-2}'(\underline{R}_{T-1} - \underline{i}_N R_{T-1}^f) + R_{T-1}^f\right)$$
$$c_{T-2}, \alpha_{T-2} \in \mathcal{F}(s_{T-2})$$

$\Rightarrow$

$$V_{T-2}(W_{T-2}, s_{T-2})$$
$$= \max_{c_{T-2}\ \alpha_{T-2}} \left\{ E\left[u(c_{T-2}) + \delta\, V_{T-1}(W_{T-1}, s_{T-1}) \mid s_{T-2} \right] \right\}$$
$$\text{st } W_{T-1} = (W_{T-2} - c_{T-2})\left(\alpha_{T-2}'(\underline{R}_{T-1} - \underline{i}_N R_{T-1}^f) + R_{T-1}^f\right)$$
$$c_{T-2}\ \alpha_{T-2} \in \mathcal{F}(s_{T-2})$$

4 holds for any $\tau = t, \quad t+1$ & so

$$V_\tau(W_\tau, s_\tau)$$
$$= \max_{c_\tau, \alpha_\tau} \overbrace{\left\{ E\left[u(c_\tau) + \delta\, V_{\tau+1}(W_{\tau+1}, s_{\tau+1}) \mid s_\tau \right] \right\}}^{L(c_\tau, \alpha_\tau; W_\tau, s_\tau)} \quad (1)$$
$$\text{st } W_{\tau+1} = (W_\tau - c_\tau)\underbrace{\left(\alpha_\tau'(\underline{R}_{\tau+1} - \underline{i}_N R_{\tau+1}^f) + R_{\tau+1}^f\right)}_{R_{\tau+1}^W} \quad (2)$$
$$c_\tau, \alpha_\tau \in \mathcal{F}(s_\tau)$$
$$\& \ V_T(\cdot) = u(\cdot)$$

5 can show that if $u(\cdot)$ is increasing & concave then $V_\tau(W_\tau, s_\tau)$ is increasing & concave in W_τ

(11)

· envelope condition

1 can show $u'(c_\tau^*(W_\tau, s_\tau)) = V_{\tau W}(W_\tau, s_\tau)$

where $\left\{\begin{matrix} c_\tau^*(\cdot,\cdot) \\ \& \ \alpha_\tau^*(\cdot,\cdot) \end{matrix}\right\}$ is the soln to (1)

2 proof

foc for (1)

$$\frac{\partial L(c_\tau^*(W_\tau, s_\tau), \alpha_\tau^*(W_\tau, s_\tau); W_\tau\, s_\tau)}{\partial c_\tau} = 0$$

$$u'(c_\tau^*(W_\tau, s_\tau)) - \delta E\left[V_{\tau+1\, W}(W_{\tau+1}, s_{\tau+1}) R_{\tau+1}^w \mid s_\tau\right] = 0$$

$$\frac{\partial L(c_\tau^*(W_\tau, s_\tau)\ \alpha_\tau^*(W_\tau, s_\tau); W_\tau\, s_\tau)}{\partial \alpha_\tau} = 0$$

$$E\left[V_{\tau+1\, W}(W_{\tau+1}, s_{\tau+1})(\underline{R}_{\tau+1} - R_{\tau+1}^f) \mid s_\tau\right] = \underline{0}$$

now

$$V_{\tau W}(W_\tau, s_\tau) = \frac{d\, L(c_\tau^*(W_\tau, s_\tau), \alpha_\tau^*(W_\tau, s_\tau); W_\tau, s_\tau)}{d W_\tau}$$

$$= \frac{\partial L(\cdot)}{\partial W_\tau} + \underbrace{\frac{\partial L(\cdot)}{\partial c_\tau}}_{=0} \frac{\partial c^*}{\partial W_\tau} + \underbrace{\frac{\partial L(\cdot)}{\partial \alpha_\tau}}_{=0} \frac{\partial \alpha^*}{\partial W_\tau}$$

$$= \delta E\left[V_{\tau+1\, W}(W_{\tau+1}, s_{\tau+1}) R_{\tau+1}^w \mid s_\tau\right]$$

so

$$u'(c_\tau^*(W_\tau, s_\tau)) = V_{\tau W}(W_\tau, s_\tau)$$

(12)

· infinite-lived

1. problem at time t

$$V_t(W_t, S_\tau) = \max E\left[\sum_{\tau=0}^{\infty} \delta^\tau u(c_{t+\tau}) \mid S_t\right]$$

s.t.

problem at time t+1

$$V_{t+1}(W_{t+1}, S_{t+1}) = \max E\left[\sum_{\tau=0}^{\infty} \delta^\tau u(c_{t+1+\tau}) \mid S_{t+1}\right]$$

st

2 problem at t looks the same as problem at t+1 so if soln to RHS is finite

Belman equation becomes

$$V(W_\tau, S_\tau) = \max_{c_\tau \alpha_\tau} \left\{E\left[u(c_\tau) + \delta V(W_{\tau+1}, S_{\tau+1}) \mid S_\tau\right]\right\}$$

st (2)

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