A BINOMIAL MODEL OF ASSET AND OPTION PRICING WITH HETEROGENEOUS BELIEFS

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ABSTRACT. This paper provides a theoretical framework for pricing assets in a multi-period economy with heterogeneous beliefs. The stock price dynamics follow a binomial lattice structure. Investors are assumed to agree on the probability, but differ in their beliefs of the asset return in each state of nature. By static and dynamic analysis, we show that the consensus belief of the market as a representative agent is a wealth weighted average of agents' subjective beliefs. Heterogeneous beliefs and speculative behavior can be used to explain anomalies such as the Risk Premium Puzzle, Risk-free rate Puzzle and the volatility skew.

JEL Classification: G12, D84.

Keywords: asset prices; heterogeneous beliefs; Trees;

1. INTRODUCTION

Since its advent in the 1970s, binomial models have been popular and widely used in the finance literature. The binomial model was first proposed by Cox, Ross and Rubinstein (1979) (CRR) which have subsequently become one of the most cited paper in the finance literature. At the time of its publication, economists were not conversant with

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the mathematical tools used to derive the Black-Scholes option pricing formula, so the CRR paper re-derived the formula as a limit from the binomial model. Essentially, the binomial pricing model uses a “discrete-time framework” to model the dynamics of the underlying stock price using a binomial lattice (tree), the larger the number of steps taken, the smaller the intermediate time interval and thus higher the frequency.

Academic and practitioners have since used binomial models for valuing more exotic and path-dependent options, interest rate term modelling and for other purposes (see Hoek and Elliot (2006) for a summary). A common assumption amongst these binomial models is that all investors have the perfect rational expectation of the future stock returns in the up and down state. However, this assumption is far from reality. The literature has seen increasing evidence on heterogeneity and bounded rationality, that is investors hold different beliefs and make optimal investment decisions in accordance with their beliefs. Heterogeneous beliefs could be caused by different information or different interpretation of the same information. Moreover, it is well known that traditional finance theory based on homogeneity and rational expectation has difficulties to explain some well known puzzles in the finance literature such as the Risk Premium Puzzle, Risk-free rate puzzle and the volatility skew of implied volatilities of option prices. The above shortcomings of homogeneous rational expectation have led to the recent literature on asset pricing under heterogeneous beliefs. It is a difficult task to incorporate this aspect of reality into financial modelling because one must now model each agent’ investment behavior and his/her learning process. This typically results in nonlinear dynamical systems. Traditionally, people worked within the continuous time framework and use Bayesian updating or filtering techniques to model investors’ learning process, see William (1977), Detemple and Murthy (1994), David (2008) and Jouni and Napp (2007). These models show that speculative behavior of individual investors, which are not reflected in the aggregate data, can increase the risk premium of the stock and reduce the risk-free rate. These models typically use complicated mathematical tools in stochastic calculus and filtering theory, however it is not likely that all market participants are familiar with these tools and use them to make portfolio decisions. This might be the reason why heterogeneous agent models (HAM) became popular in recent years, these models assume that investors uses different simple rules of thumb in making portfolio decisions and then asset price dynamics are results from interactions of heterogeneous agents with different attitudes towards risk and different expectations about the future evolution of asset prices. One of the key aspects of these models which distinguish them from the previous literature is the feedback of expectations - agents' decisions are based upon predictions of future values of endogenous variables whose actual values are determined in equilibrium. In particular, Brock and Hommes (1997, 1998) proposed an *Adaptive Belief System* model of economic and financial markets. Agents adapt their beliefs over time by choosing from different predictors or expectations functions, based upon their past performance. The resulting nonlinear dynamical system is capable of generating a wide range of complex price behavior from local stability to high order cycles and chaos. Hommes (2006), LeBaron (2006) and Chiarella, Deici and He (2008) provide a comprehensive survey of the development of HAMs. A well known short-coming of the HAMs is that they typically have too many free variables, so it is hard to identify which parameter is responsible for the stylized fact of the simulated asset dynamics.

This paper tries to overcome the short-comings of the above models by considering a multi-period binomial model with one risky and one risk-free asset. We assume that all investors have log-utility who tries to maximize their terminal wealth, they agree on the probability, but differ in their beliefs of the asset return in each state of nature. Investors are bounded rational in the sense that they make their optimal portfolio decision based on their subjective beliefs. Then we provide a method to aggregate heterogeneous beliefs on a binomial lattice by introducing the concept of a consensus belief. The consensus belief represents the perspective of the aggregate market as a composite investor. We show that the consensus belief is basically a weighted average of investors' subjective beliefs, weights are determined by individual investors' wealth as a percentage of the total market wealth. We then use this framework to price assets and contingent claims under heterogeneous beliefs and compare them to the standard results in homogeneous case. We show that heterogeneous beliefs and speculative behavior can be used to explain anomalies such as the Risk Premium Puzzle, Risk-free rate Puzzle and the volatility skew. The model is standard except that investors are assumed to have heterogeneous beliefs regarding the next period's stock return in the up and down states respectively. Therefore, any of our results deviating from the standard results under homogeneous beliefs can only be caused by the heterogeneity amongst investors' beliefs. Furthermore, the concept of a consensus belief allows us to relate the heterogeneous market to an equivalent homogeneous market. This means that we can use many of the theoretical results already available.

Our paper is organized as follows. Section 2 presents the setup of the economy. Section 3 defines and characterizes the consensus belief and shows how consensus belief leads to the construction of a representative agent in the market. Section 4 performs static analysis and studies the impact of heterogeneous beliefs on the equilibrium price of the risky asset in a single-period setting. Section 5 discusses the relationships between heterogeneous beliefs and Risk Premium Puzzle and the Risk-free rate puzzle. Section 6 studies the dynamic impact of heterogeneous beliefs on the simulated time series of equilibrium risk-free rate, equity premium and investors' wealth share. Section 7 discusses the relationship between heterogeneity and option prices. Section 8 concludes.

2. SETUP OF THE ECONOMY AND PORTFOLIO STRATEGIES

We consider a rather simple economy with one risky and one riskless asset. Let time be discrete and finite, index by $t = 0, 1, 2, \cdots, T$. The risky asset has one share available and the riskless asset is in net zero supply for all time t. There are I investors in the economy, indexed by $i = 1, 2, \cdots, I$. Investor i's objective at time t is to maximize the quantity

$$\mathbb{E}_t^i\Big(U(\tilde{W}_i(T))\Big) \tag{2.1}$$

where $\mathbb{E}_t^i$ denotes investor i's expectation of the outcome of the market at time T conditional on the information available and him/her belief at time t. Furthermore, $U(\cdot)$ is investor i's utility function and $\tilde{W}_i(T)$ is his/her portfolio's terminal wealth at time T which is random.

(H1) *Assume* $U(x) = \ln(x)$.

(H2) *Assume that stock price follow a multi-period Cox-Ross-Rubinstein model. This means, given information at time* t*, the price of the risky asset at time* $t+1$ *has the following probability distribution,*

$$S(t+1) = \begin{cases} S(t)\ u(t+1), & \text{p;} \\ S(t)\ d(t+1), & \text{1-p.} \end{cases}$$

with $d(t) < 1 + r_f(t) < u(t)$ *where* $r_f(t)$ *is the return of the riskless asset over the period* $[t, t+1]$.

(H3) *Assume that all investors agree on the probabilities associated with each state, however have their own subjective beliefs of the rate of return at each state. Let $u_i(t+1)$ and $d_i(t+1)$ denote investor i's belief of $u(t+1)$ and $d(t+1)$, respectively. $\mathcal{B}_i(t) := (u_i(t+1), d_i(t+1))$ denotes investor i's set of subjective belief of $S(t+1)$ at time t.*

Let $\omega_i(t)$ be the proportion of investor i's wealth $W_i(t)$, at time t, invested in the risky asset and define the future return of the risky asset as

$$r(t+1) = \frac{S(t+1) - S(t)}{S(t)}$$

which is random at time t. Then, given (H1) to (H3), investor i's objective in equation (2.1) becomes

$$\max_{\{\omega_i(t),\omega_i(t+1),\cdots,\omega_i(T-1)\}} \ln(W_i(t)) + \sum_{s=t}^{T-1} \mathbb{E}_t^i \left[\ln \left(1 + r_f(s+1) + \omega_i(s)(r(s+1) - r_f(s+1)) \right) \right] \tag{2.2}$$

The optimization problem in (2.2) can be solved using *dynamic programming* or the *Martingale Approach*. Detailed solution to the problem under both methods can be found in Cvitanic and Zapatero (2004) Chapter 4.

Lemma 2.1. *Let $\tilde{u}(t+1) = u(t+1) - (1 + r_f(t))$ and $\tilde{d}(t+1) = d(t+1) - (1 + r_f(t))$ be the excess rate of return in the up and down states, respectively over the period $[t, t+1]$. The solution to investor i's multi-period optimization problem in equation (2.2) is given by*

$$\hat{\omega}_i(t) = (1 + r_f(t+1)) \frac{\tilde{u}_i(t+1)\ p + \tilde{d}_i(t+1)\ (1-p)}{-\tilde{u}_i(t+1)\ \tilde{d}_i(t+1)} \tag{2.3}$$

The intuition is that maximizing the logarithm of a portfolio's terminal wealth is equivalent to maximizing the expected growth rate $\mathbb{E}[\ln(1 + R(t+1))]$ period by period, where $R(t+1)$ is portfolio's rate of return from t to $t+1$. This is so called the short-sighted or *myopic behavior* of logarithmic utility, because log-utility maximizers do not consider any future investment opportunities in their portfolio selections, see Cvitanic and Zapatero (2004) chapter 4.

3. Representative agent, Consensus belief and Market equilibrium

In this section, we first introduce a consensus belief. Then we show how the consensus belief can be constructed in our economy. Since we have one risky asset available in our market and zero net supply for the riskless asset, in order for the market to clear, investors' total dollar demand for the risky asset must equal to the aggregate market wealth. This means that in equilibrium the price of the risky asset must equal to the aggregate market wealth in equilibrium, at each time t, that is,

$$\sum_{i=1}^{I} \hat{\omega}_i(t) \; W_i(t) = W_m(t) = S(t), \tag{3.1}$$

where $W_m(t) = \sum_{i=1}^{I} W_i(t)$ denote the aggregate market wealth at time t. We refer to equation (3.1) as the *market clearing condition* for our economy at each time t. Substituting equation (2.3) into the market clearing condition in (3.1) leads to the following expression for the equilibrium risk free rate from time t to $t+1$,

$$r_f(t+1) = W_m(t) \; \left(\sum_{i=1}^{I} \frac{\tilde{u}_i(t+1) \; p + \tilde{d}_i(t+1) \; (1-p)}{-\tilde{u}_i(t+1) \; \tilde{d}_i(t+1)} W_i(t) \right)^{-1} - 1. \tag{3.2}$$

Although it is clear that market is in equilibrium if the risk free rate $r_f(t)$ is set to the appropriate level according to equation (3.2)[1], it is not clear what the market's belief of the future outcome at time $t+1$ is and how the market belief is related to the heterogeneous beliefs. We tackle this problem by introducing the concept of a *consensus belief*.

[1]The expression in equation (3.2) is implicit since both $\tilde{u}_i(t)$ and $\tilde{d}_i(t)$ depend on $r_f(t)$.

Definition 3.1. *A belief $\mathcal{B}_m(t) = (u_m(t+1), d_m(t+1))$, defined by the return of the risky asset in the up and down state respectively at time $t+1$, is called a* **consensus belief** *at time t if the market equilibrium price for the risky asset and risk free rate under the heterogeneous beliefs is also those under the homogeneous belief $\mathcal{B}_m(t)$.*

The introduction of a consensus belief allows the transformation of the a market with heterogeneous beliefs to a market under which all agents are identical in their beliefs. If the aggregate market invests as a sole log-utility maximizer, its belief at time t coincide with the consensus belief $\mathcal{B}_m(t)$ which if exists also represents the market's belief. Intuitively, this is the most informed belief of the future because it takes into account every individual investor's subjective belief at time t. Now we show by construction that such a consensus belief exists in our particular economy.

Proposition 3.2. *Under assumptions (H1) to (H3),*

(i) *The consensus belief at time t, $\mathcal{B}_m(t)$, is given by*

$$u_m(t+1) = \tilde{u}_m(t+1) + (1+r_f)$$
$$d_m(t+1) = \tilde{d}_m(t+1) + (1+r_f) \tag{3.3}$$

where

$$\tilde{u}_m(t+1) = \left(\sum_{i=1}^{I} w_i(t) \;\; \tilde{u}_i(t+1)^{-1} \right)^{-1}, \tag{3.4}$$

$$\tilde{d}_m(t+1) = \left(\sum_{i=1}^{I} w_i(t) \;\; \tilde{d}_i(t+1)^{-1} \right)^{-1} \tag{3.5}$$

and $w_i(t) = \frac{W_i(t)}{W_m(t)}$ is the wealth share of investor i at time t.

(ii) *The equilibrium risk free rate is given by*

$$r_f(t+1) = \left(\frac{\tilde{u}_m(t+1) \;\; p + \tilde{d}_m(t+1) \;\; (1-p)}{-\tilde{u}_m(t+1) \;\; \tilde{d}_m(t+1)} \right)^{-1} - 1, \tag{3.6}$$

or it can be rewritten as

$$r_f(t+1) = \frac{-\tilde{u}_m(t+1)\ \tilde{d}_m(t+1)}{\mathbb{E}_t^m(\tilde{r}(t+1)) - r_f(t+1)} - 1. \tag{3.7}$$

(iii) *The market consensus belief of the state prices or the risk neutral probabilities of the up and down state at time t are given by*

$$q_u(t) = \frac{-\tilde{d}_m(t+1)}{\tilde{u}_m(t+1) - \tilde{d}_m(t+1)}, \tag{3.8}$$

$$q_d(t) = \frac{\tilde{u}_m(t+1)}{\tilde{u}_m(t+1) - \tilde{d}_m(t+1)}. \tag{3.9}$$

(iv) *In equilibrium, the stock price at time t is given by*

$$S(t) = \frac{\mathbb{E}_t^{Q_m}(S(t+1))}{1+r_f(t+1)} = \frac{\mathbb{E}_t^m(Z(t+1)S(t+1))}{1+r_f(t+1)}, \tag{3.10}$$

where

$$Z(t+1) = \begin{cases} \frac{q_u(t)}{p}, & p; \\ \frac{q_d(t)}{1-p}, & 1\text{-}p \end{cases} \tag{3.11}$$

is the Randon-Nikodym derivative that change the probability measure from m to Q_m, and often referred to as the "pricing kernel" in the asset pricing literatures.

Proof of Proposition 3.2 is given in the appendix. It shows that the market consensus belief of the excess return in the up and down states are simply wealth weighted harmonic means of their correspondents under individual investors' beliefs. Equation (3.7) shows that the risk-free rate is negatively related to the (i) equity risk premium and positively correlated with (ii) volatility of the equity premium under the market's belief. This might be at first sight counter-intuitive since increase in (i) and decrease in (ii) would usually be incentives for an investor to allocate more wealth to the risky asset and this is in fact the case according to equation (2.3). Then the question is: should not the risk-free rate be higher as a result?

Now suppose that investor i's belief about the future excess return is independent of the current level of the risk-free rate, that is $(\tilde{u}_i(t+1), \tilde{d}_i(t+1))$ are predetermined, then his/her portfolio weight in the risky asset is actually positively related to the risk-free rate r_f according to equation (2.3). The intuition is that if an investor has already made up his/her mind about the level of risk premium in the market, then a higher risk-free rate will only imply a higher expected return for the stock. Hence the investor will allocate even more wealth to the stock in order to maximize portfolio growth.

Furthermore, since the set $\{w_i(t)\}$ actually satisfies the criterion of a probability measure, we can rewrite the consensus belief as

$$\tilde{u}_m(t+1)^{-1} = \mathbb{E}_W(\tilde{u}(t+1)^{-1}), \tag{3.12}$$

$$\tilde{d}_m(t+1)^{-1} = \mathbb{E}_W(\tilde{d}(t+1)^{-1}), \tag{3.13}$$

where $\mathbb{E}_W$ denote the expectation under the probability measure $\{w_i(t)\}$. Moreover, it is clear that the market's belief $\mathcal{B}_m(t)$ and also the state prices $(q_u(t),\, q_d(t))$ evolve stochastically over time even if each investor's belief $\mathcal{B}_i(t)$ remains constant over the time. This is because investors' wealth shares $w_i(t)$ are stochastic.

4. IMPACT OF HETEROGENEOUS BELIEFS

In this section, we examine the impact of heterogeneous beliefs on the market consensus belief. First consider the effect of the change in a particular investor i's belief on the consensus belief. From equation (3.4) and (3.5), we can calculate the following first and second order partial derivatives of the consensus belief w.r.t to a particular investor's subjective belief. For the up state we obtain

$$\frac{\partial \tilde{u}_m(t+1)}{\partial \tilde{u}_i(t+1)} = w_i(t) \left(\frac{\tilde{u}_m(t+1)}{\tilde{u}_i(t+1)}\right)^2 \tag{4.1}$$

$$\frac{\partial^2 \tilde{u}_m(t+1)}{\partial \tilde{u}_i(t+1)^2} = 2\frac{W_i(t)}{W_m(t)} \frac{\tilde{u}_m(t+1)}{\tilde{u}_i(t+1)^2}\left(\frac{W_i(t)}{W_m(t)} \frac{\tilde{u}_m(t+1)}{\tilde{u}_i(t+1)} - 1\right). \tag{4.2}$$

Similarly for the down state we obtain the following,

$$\frac{\partial \tilde{d}_m(t+1)}{\partial \tilde{d}_i(t+1)} = w_i(t) \left(\frac{\tilde{d}_m(t+1)}{\tilde{d}_i(t+1)}\right)^2 \tag{4.3}$$

$$\frac{\partial^2 \tilde{d}_m(t+1)}{\partial \tilde{d}_i(t+1)^2} = 2\frac{W_i(t)}{W_m(t)} \frac{-\tilde{d}_m(t+1)}{\tilde{d}_i(t+1)^2}\left(1 - \frac{W_i(t)}{W_m(t)} \frac{\tilde{d}_m(t+1)}{\tilde{d}_i(t+1)}\right). \tag{4.4}$$

It can be seen from equations (4.1) and (4.2) that $\tilde{u}_m(t+1)$ and $\tilde{d}_m(t+1)$ are positively correlated with $\tilde{u}_i(t+1)$ and $\tilde{d}_i(t+1)$ respectively. The sensitivity is higher for a individual with a larger initial wealth $W_i(t)$. Moreover, if we assume that $W_i(t) \ll W_m(t)$, that is each investor has a very small wealth share, then according to equations (4.3) and (4.4) sensitivity is likely to be negatively related to investor beliefs in the upstate and positively related in the downstate. This analysis indicates, individual investors' beliefs have a positive impact on the market consensus belief. For the upstate, a wealthier and relatively pessimistic investor will have a stronger impact on the market belief while for the downstate, a wealthier and relatively optimistic investor will have a stronger impact.

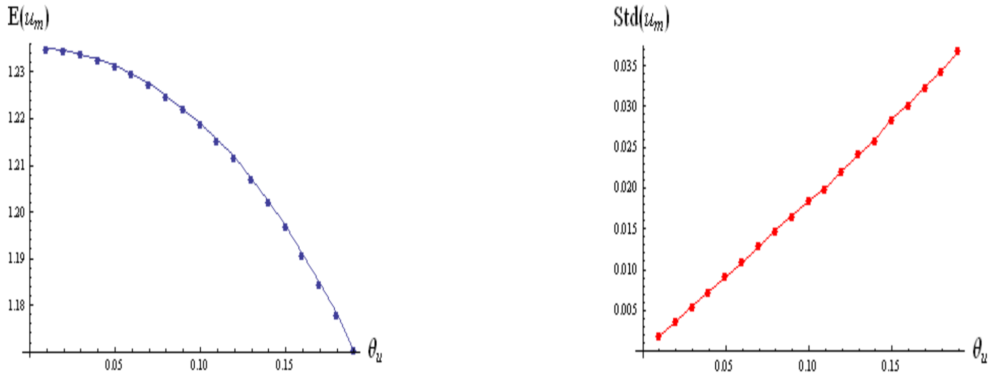
The following discussion is devoted to Miller's hypothesis (Miller (1977)) that divergence of opinion about market's future return amongst the investors results in a lower expected rate of return for the market compare with an otherwise similar situation. To see whether Miller's hypothesis is true in our particular economy, we consider the following rather simple example. First, we need to define a *Mean Preserved Spread*(MPS).

Definition 4.1. *Consider a random variable X with mean μ and variance σ^2, a random variable $Y = X + \epsilon$ is a mean preserved spread of X if $\mathbb{E}(\epsilon|X) = 0$.*

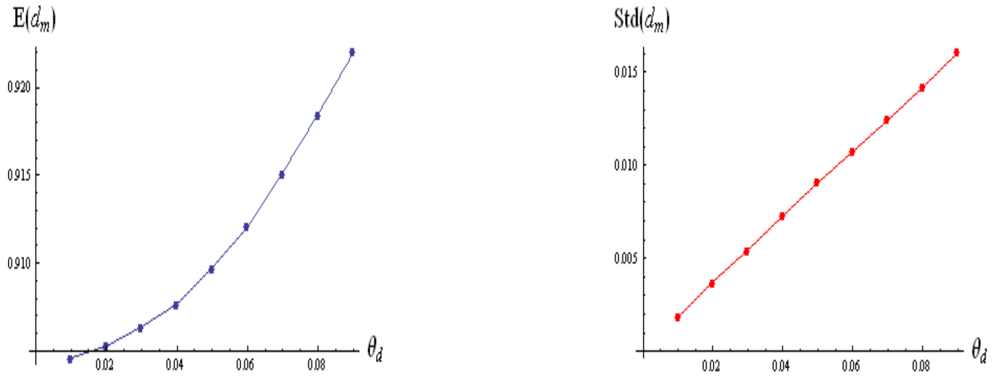
Example 4.2. *Let there be $I = N$ investors. Consider a benchmark belief $\mathcal{B}_o(t) = (u_o(t+1), d_o(t+1))$, and assume that investors' subjective beliefs diverge from the benchmark belief. We do this by imposing a* Mean Preserved Spread *(MPS) on investors' beliefs. Let investor i's belief be given by $\mathcal{B}_i(t) = (u_o(t+1) + \epsilon_{iu}, d_o(t+1) + \epsilon_{id})$ where $\epsilon_{iu} \overset{\text{iid}}{\sim} Unif(-\theta_u, \theta_u)$, similarly $\epsilon_{id} \overset{\text{iid}}{\sim} Unif(-\theta_d, \theta_d)$. Therefore investors' divergence of opinion regarding the stock returns in both up and down states are identically and independently uniformly distributed.*

In Example 4.2, it is clear that investors have heterogeneous beliefs regarding the future return of the risk asset with average belief corresponding to the benchmark belief $\mathcal{B}_o(t)$. Now the question is whether this MPS on individual investors' subjective beliefs will push up the equilibrium price for the risky asset and reduce its expected future return. To answer this question, we let $\mathcal{B}_o(t) = (1.235, 0.905)$, $p = 0.5$ and $w_i(t) = 1/N$ where $N = 10$ for all i, this means that up and down states are equal likely to occur and wealth is evenly distributed. Next, we perform Monte-Carlo simulations by varying the parameters θ_u and θ_d. The equilibrium risk-free rate r_f[2] and the consensus belief $\mathcal{B}_m(t)$ are determined using Proposition 3.2. Figure 4.1 compares the average consensus belief with the benchmark for different combinations of (θ_u, θ_d).

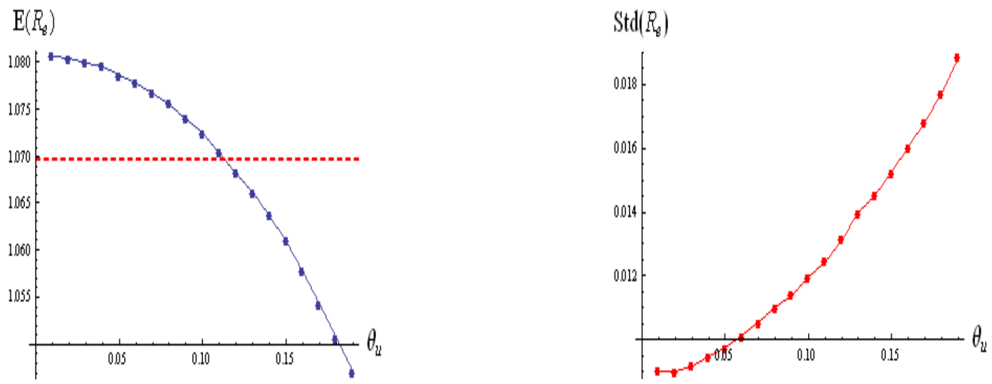
[2]The risk-free rate is solved numerically by the assumption of net-zero supply



(a1) $\theta_u \in [0.01, 0.2]$, $\theta_d = 0$



(a2) $\theta_u = 0$, $\theta_d \in [0.01, 0.1]$



(a3) $\theta_u \in [0.01, 0.2]$, $\theta_d = 0.1$

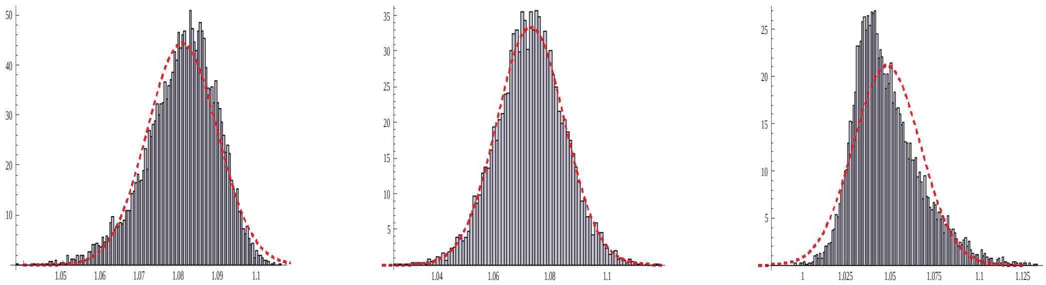
FIGURE 4.1. Impact of divergence of opinion on the expected future stock return R_e under the market consensus belief. The left panel shows the average expected return $\mathbb{E}(R_e)$ (blue line) with the red dashed line being the benchmark expected return (1.0698). The right panel shows the volatility of the expected returns $Std(R_e)$ (red line).

Figure 4.1 shows that when investors' opinions diverge farther from the benchmark stock return in the upstate (downstate), the market expects a lower (higher) return relative to the benchmark, see the left panels of Figure 4.1 (a1) and (a2). In other words, the market as a consensus investor prefers more (less) uncertainty regarding the future stock return in the downstate (upstate). The term "uncertainty" in this context refers to the measure of dispersion in investors' beliefs (θ_u, θ_d) since increase in these two parameters implies there is more uncertainty regarding the future return of the risky asset in up and downstate respectively. When the two effects are combined, the left panel of (a3) shows that the market belief of the expected future return will be below (above) the benchmark when investors' subjective beliefs are more dispersed about the future return in the upstate (downstate). Remark 4.3 summarizes our finding in Example 4.2.

Remark 4.3. *Assume investors' wealth are evenly distributed, then for the Miller's hypothesis to hold, which means the market as a consensus investor believes that divergence of opinion is negatively related to expected future stock return, it is necessary for the investors to have a greater divergence of opinions regarding the future return of the stock in the upstate than in the downstate, that is we require* $\theta_u > \theta_d$ *in Example 4.2.*

However, since the market has to clear, the market equilibrium price for the stock has to equal to the aggregate market wealth in any case, so we cannot verify whether divergence of opinion will push up the equilibrium price. One needs a multi-asset model to explain this part of the Miller's hypothesis. Furthermore, the standard deviation of u_m and d_m seem to linearly and positively related to θ_u and θ_d as indicated by the right panels of Figure 4.1. So the market becomes more uncertain about the future return of the stock in either state as investors' opinions become more divergent.

Next, it would also be interesting to see the shape of the probability distribution of the expected future stock return. In particular, how it changes with different combination of (θ_u, θ_d), and whether it is normally distributed.



(b1) $(\theta_u, \theta_d) = (0.01, 0.1)$ (b2) $(\theta_u, \theta_d) = (0.1, 0.1)$ (b3) $(\theta_u, \theta_d) = (0.2, 0.1)$

FIGURE 4.2. Impact of divergence of opinion on the distribution of expected future stock return under the market consensus belief

One can see from Figure 4.2 that the distribution of the expected future return under the consensus belief is normally distributed when divergence of opinion is equal for both states ($\theta_u = \theta_d$). It becomes high peaked and positively (negatively) skewed when there is greater divergence of opinion regarding future return in the upstate (downstate), i.e $\theta_u > \theta_d$ ($\theta_u < \theta_d$). This means that under the consensus belief, greater divergence of opinion in the upstate (downstate) on average leads to aggregate pessimism (optimism) compare to the benchmark belief, but also a higher probability of an extreme positive (negative) stock return than the standard normal distribution. In particular, when investors disagree more about future returns in the downstate, the market is on average optimistic, but it also believes there is a significant chance for a extremely low expected return, see (b1). Since there is little disagreement about $u(t+1)$, the market belief $u_m(t+1)$ would be very close to the benchmark $u_o(t+1)$, so a very low expected return in this case means that the market belief of the future return in the downstate $d_m(t+1) - 1$ is extremely negative. This phenomenon suggests that the market is crash averse when investors have heterogeneous beliefs, in particular when divergence of opinion is greater for future stock return in the downstate than the upstate.

5. HETEROGENEITY, RISK PREMIUM PUZZLE (RPP) AND THE RISK-FREE RATE PUZZLE (RFRP)

The Risk Premium Puzzle (RPP) and Risk-free rate puzzle (RFRP) are well documented by Mehra and Prescott (1985) and Weil (1989) respectively. Basically, the puzzles come form the fact that aggregate economic and financial data cannot explain the observed high equity premium and low risk-free rate using standard utility theory with a representative agent. Weil (1989) shows that the puzzles remain even if one uses a rather general utility function to represent investors' preferences. Both papers suggested that heterogeneity amongst investors may help to resolve the puzzles, because differences in subjective beliefs are not captured by aggregate data.

To demonstrate how agents' subjective beliefs about the future return of the risky asset and heterogeneity in those beliefs can have different implications on the risk premium and the risk-free rate in equilibrium, we now consider the following contrasting cases.

(i) Assume (H1) to (H3) and one agent R in the economy who at time t form belief of $S(t+1)$ in terms of its future return, i.e $\mathcal{B}_o(t) := (u_o(t+1), d_o(t+1))$, $u_o(t+1)$ and $d_o(t+1)$ are agent R's belief about $u(t+1)$ and $d(t+1)$, respectively. Assume that agent R has the equity real return data from 1889-1978 and computed the sample statistic as reported in Mehra and Prescott (1985). He/she concludes that the average annual rate of return and standard deviation of the equity market is 6.98% and 16.54%, respectively and constant for all time t. Under the binomial model, given the probability of nature

p, one can calculate the payoffs in up and down states by using the following system of simultaneous equations,

$$u_o\ p + d_o\ (1-p) = 1.0698$$
$$p\ (u_o - 1.0698)^2 + (1-p)\ (d_o - 1.0698)^2 = 0.1654^2. \tag{5.1}$$

Solving the system in equation (5.1) in terms of p yields solution plotted in Figure 5.1.

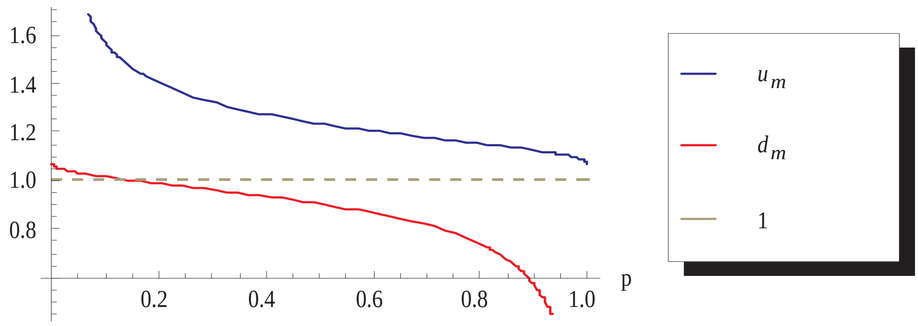


FIGURE 5.1. Solutions for u_o and d_o in terms of probability p

One can see in Figure 5.1 that u_o (d_o) decreases (increases) as p increases. This is because the upstate become more and more likely to occur, the market only require a relatively small return in the upstate to match the expected return, and a large negative return in the downstate to match the observed volatility. Next, the risk-free rate r_f can be solved *implicitly* using the market clearing condition $\hat{\omega}_m = 1$, where

$$\hat{\omega}_m = \frac{(1+r_f)((u_o - r_f - 1)\ p + (d_o - r_f - 1)\ (1-p))}{(u_o - r_f - 1)\ (d_o - r_f - 1)}.$$

The equilibrium interest rate r_f and the equity risk premium r_e as functions of the probability p is plotted in Figure 5.2.

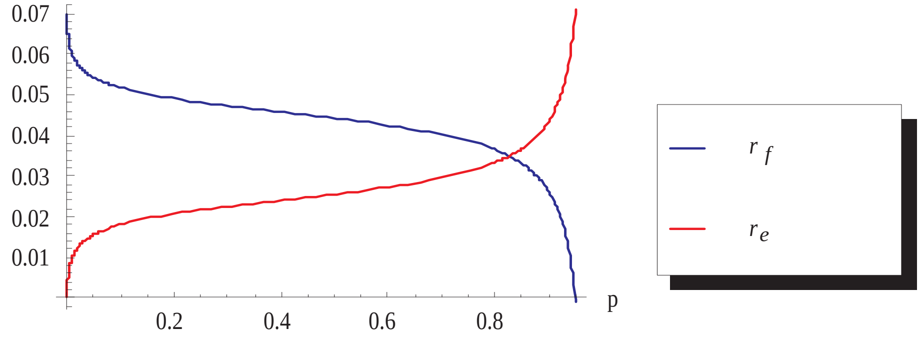


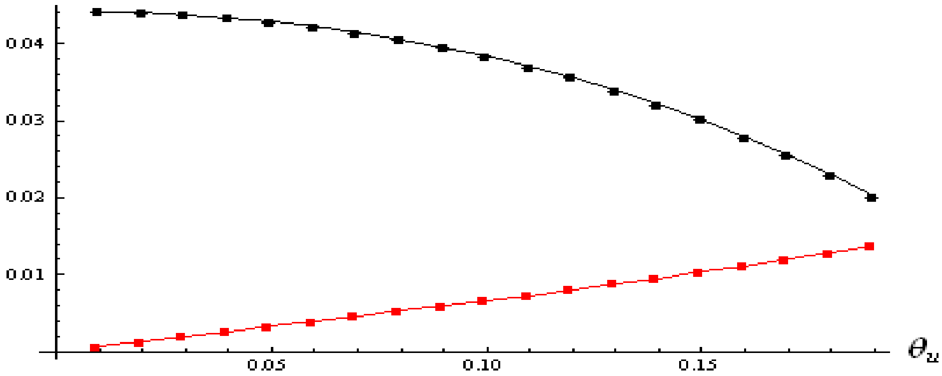
FIGURE 5.2. Solutions for r_f and r_e in terms of probability p

Figure 5.2 shows that in order to explain the observed level of risk premium, volatility and risk-free rate in the market, one needs to assume a implausible high probability for the upstate to occur in the next period, close to 95%. If one assumes that either state is likely to occur ($p = 0.5$), then one will obtain a constant risk-free rate of $r_f = 4.42\%$ instead of the observed risk-free rate with mean 0.80% and standard deviation 5.67%, see Mehra and Prescott (1985). This also implies an expected risk premium of 2.56% instead of the observed average risk premium of 6.18%. Therefore, one can see that the risk premium (risk-free rate) is on average too low (high) compare with the observed data. This is essentially the RRP and RFRP although we do not consider consumption in our model.

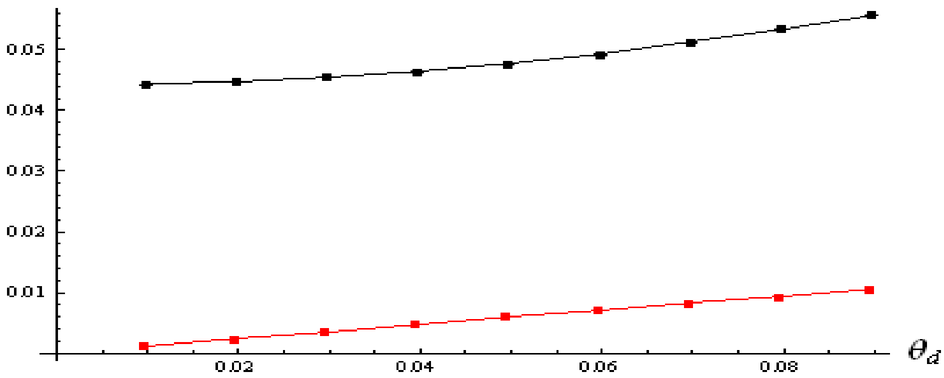
(ii) Now if we consider the cases with multi-agents similar to the setup in Example 4.2. That is agents disagree about the values $u(t+1)$ and $d(t+1)$ in the form of MPS around the benchmark belief $\mathcal{B}_o(t+1)$. It would be interesting to see whether the divergence of agents' subjective beliefs can help resolving the puzzles. As it was seen in Example 4.2, when the divergence of opinion is greater regarding future return in the upstate than in the downstate, i.e $\theta_u > \theta_d$, the market on aggregate becomes pessimistic about the expected future stock return relative to the benchmark average belief. This intuitively means that the aggregate market would be less willing to allocate wealth to the stock and more willing to invest in the risk-free asset which should drive down the risk-free rate. However, the market also becomes on average overconfident relative to the benchmark. Because u_m (d_m) is expected to increase (decrease) with greater divergence of opinion, so if we define the volatility of future stock return under the market consensus and the benchmark belief respectively as $\sigma_m = (p\ u_m^2 + (1-p)\ d_m^2)^{\frac{1}{2}}$ and $\sigma_o = (p\ u_o^2 + (1-p)\ d_o^2)^{\frac{1}{2}}$, then σ_m would be small than σ_o. This should cause the aggregate market to be more willing to invest in the stock and thus drive up the risk-free rate. The question is in equilibrium, which effect (pessimism or overconfidence) is going to dominate the market consensus belief, and thus have a greater impact on the risk-free rate.

Figure 5.3 shows that θ_u (θ_d) is negatively (positively) correlated with the average equilibrium risk-free rate respectively and both parameters are positively correlated with the volatility of the risk-free rate. The reason is that the aggregate market as a consensus investor become pessimistic (optimistic) and less (more) willing to invest in the risky asset when there is a divergence of opinions regarding the future return in the upstate (downstate), see Fig. 5.3 (b1) and (b2). When the two effects are combined, one observes from Fig. 5.3 (b3) that it is necessary for θ_u to exceed θ_d by a significant amount, in this case 5%, in order for the aggregate pessimism to dominate the aggregate overconfidence and thus drive down the average equilibrium risk-free rate below the benchmark risk-free rate.

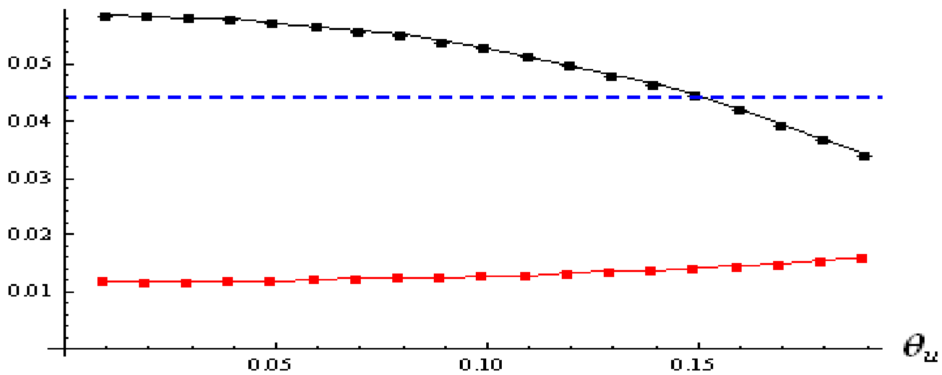
Now we can look at the actual distribution of the risk-free rate with different level of divergence of opinion. Figure 5.4 shows that the distribution is positively (negatively) skewed when θ_u (θ_d) is relatively higher, see Fig. 5.4 (c1) and (c3), and relatively more symmetrically distributed when $\theta_u = \theta_d$, see Fig. 5.4 (c2). This suggests that



(b1) $\theta_u \in [0.01, 0.2], \theta_d = 0$



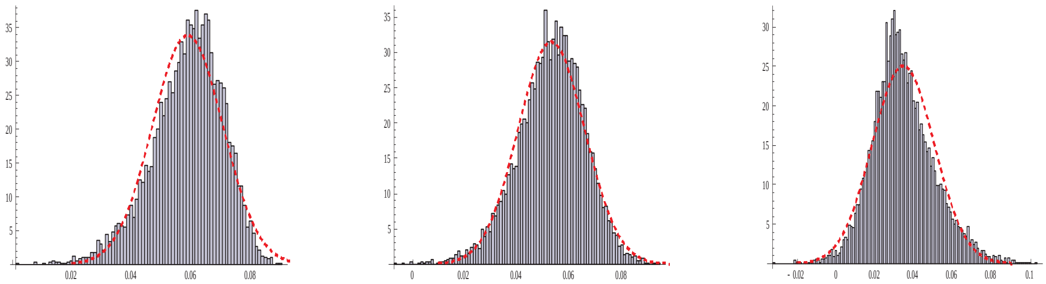
(b2) $\theta_u = 0, \theta_d \in [0.01, 0.1]$



(b3) $\theta_u \in [0.01, 0.2], \theta_d = 0.1$

FIGURE 5.3. Impact of divergence of opinion on the expected(black line) and volatility (red line) of the equilibrium risk-free rate. The blue dashed line represent the level of the equilibrium risk-free rate under the benchmark belief (4.42%).

greater divergence regarding future stock return in the upstate (downstate) leads to lower (higher) interest rate on average, but also a higher probability of extremely high (low) risk-free rate comparing to the normal distribution. The intuition for this result follows from the example in the previous section. As we know from in Figure 4.2 (b1), greater divergence regarding future stock return in the downstate increases the chance of a market crash under the market consensus belief, in which case the aggregate market would invest more into the safe asset and thus lower the risk-free rate.



(c1) $(\theta_u, \theta_d) = (0.01, 0.1)$ (c2) $(\theta_u, \theta_d) = (0.1, 0.1)$ (c3) $(\theta_u, \theta_d) = (0.2, 0.1)$

FIGURE 5.4. Impact of divergence of opinion on the distribution of equilibrium risk-free rate

In summary, this sections shows that when investors' subjective beliefs diverge symmetrically and uniformly from a benchmark belief, the aggregate market as a consensus investor becomes on average pessimistic and overconfident about the future stock return relative to the benchmark. Moreover, in order for the average equilibrium risk-free rate to be lower than the one implied by the benchmark belief, it is necessary for the investors to have a significantly greater divergence of opinions regarding the future stock return in the upstate than in the downstate, ensuring that aggregate pessimism will dominate the aggregate overconfidence. Therefore, if the benchmark belief represent an average belief in the market which is calculated using the observed stock return data, then heterogeneity in agents' beliefs provide a possible explanation for observed low average equilibrium risk-free rate.

6. Dynamic Impact of Heterogeneous beliefs

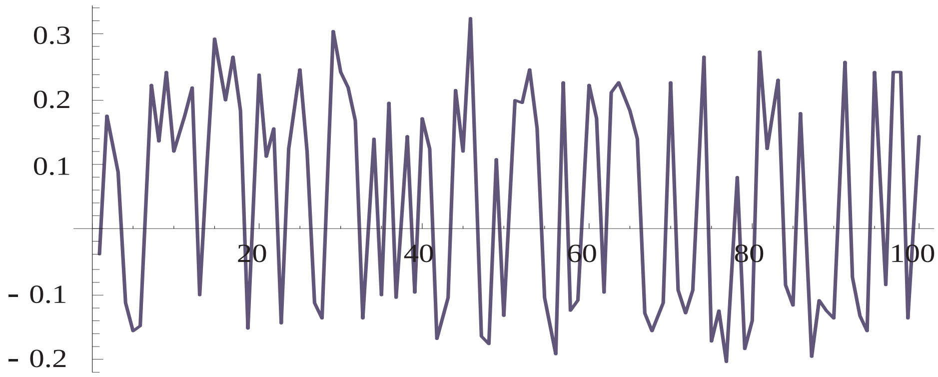
In the last section, we have made an attempt to explain the Risk Premium and the Risk-free rate Puzzles using divergence of beliefs of future excess returns among market participants. However, all the analysis were fixed at one point of time t, so it can only be considered as a static analysis. In this section, we want to examine the impact of heterogeneous beliefs in a dynamic setting. This is possible because our model is multi-period though investors make portfolio decisions of the same form at each time t, so one can also think of this situation as a repeated one-period dynamic model. By imposing various structure to the model, we examine the time series with different properties for the risk-free rate and the risk premium.

Example 6.1. *In this example, we consider Example 4.2 in a dynamic setting. We assume that true future stock return at any time t in eiher up or down state is uniformly distributed among individual subjective beliefs. Moreover, θ_u and θ_d are constant and exogenously given. Furthermore, agents revise their belief every period, they do so randomly following a iid uniform distribution (MPS), i.e $u_i(t) \stackrel{\text{iid}}{\sim} Unif(-\theta_u + u_o(t), \theta_u + u_o(t))$ and $d_i(t) \stackrel{\text{iid}}{\sim} Unif(-\theta_d + d_o(t), \theta_d + d_o(t))$ for $i = 1, 2, \cdots, I$. $u_o(t)$ and $d_o(t)$ are as specified in Example 4.2. Initially, the aggregate market wealth is 1 unit and is evenly distributed among the investors.*

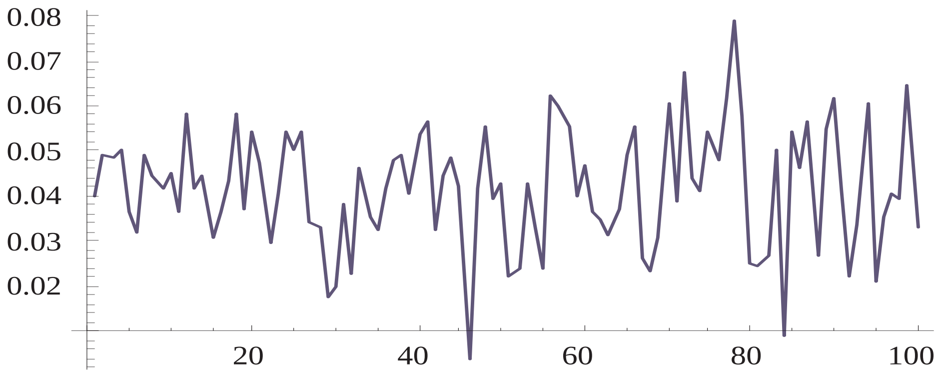
In Example 6.1, agents are in constant disagreement and do not engage in any learning process. This is because any learning attempt will not enhance agents' portfolio performance and taking random "guesses" is the best they can do, since everyone has an equal chance of being correct with their beliefs. We ran the simulation 100 periods with $\theta_u = 0.1$ and $\theta_d = 0.05$. Figure 6.1 shows resulting time series for the risk-free rate $r_f(t)$ and the risk premium $r(t) - r_f(t)$ together with the wealth share of individual agents $w_i(t)$. The risk-premium and the risk-free rate process all looks to be fluctuating around a mean, while the individual agents' wealth shares appear to evolve quite randomly over time. The reason for this is that none of the agents is able to systematically outperform other agents in the market, thus dominate the market consensus belief. At each period, both the subjective beliefs and the true market outcome in either up or downstate are random and uniformly distributed. Therefore, the risk-free rate tends to mean-revert back to its mean quite quickly without any significant patterns.

Example 6.2. *In this example, we consider the same case as Example 6.1. Except that agents do not revise their belief every period, instead they hold on to their initial belief, which follows a iid uniform distribution (MPS) about the benchmark belief, i.e* $u_i(0) \overset{\text{iid}}{\sim} Unif(-\theta_u + u_o(t), \theta_u + u_o(t))$ *and* $d_i(0) \overset{\text{iid}}{\sim} Unif(-\theta_d + d_o(t), \theta_d + d_o(t))$ *for* $i = 1, 2, \cdots, I$*, then* $u_i(t) = u_i(0)$*,* $d_i(t) = d_i(0)$ *for all* $t > 0$*.*

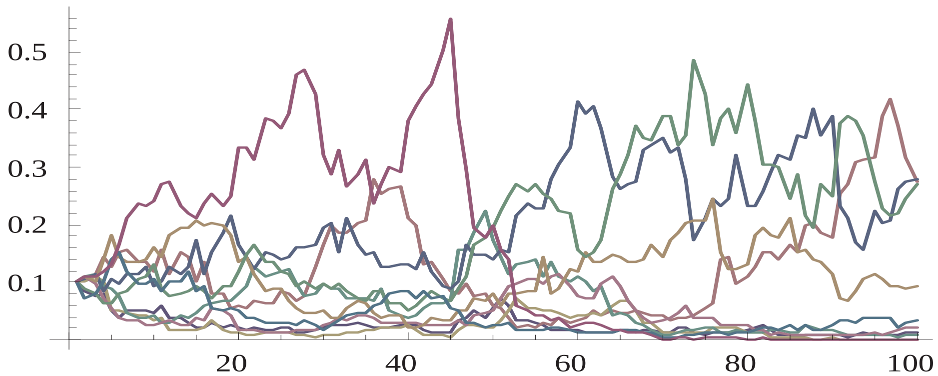
In Example 6.2, agents are again in constant disagreement and do not engage in any learning process. But now they do not even update their "guesses" about the future return. Intuitively, revising or not revising does not affect agents' future wealth since any "guess" is just as good as the other. We ran the simulation 100 periods with $\theta_u = 0.1$ and $\theta_d = 0.05$. Figure 6.2 shows resulting time series for the risk-free rate $r_f(t)$ and the risk premium $r(t) - r_f(t)$ together with the wealth share of individual agents $w_i(t)$. Comparing plots Fig. 6.2 (e3) with Fig. 6.1 (d3) and plot Fig. 6.2 (e2) with Fig. 6.1 (d2), one notices that the resulting time series are quite different. Firstly, the wealth share processes of individual investors tend to form several groups, this is because investors are stuck with their initial belief, so investors who have similar beliefs initially will also have similar wealth share over time. This means the risk-free rate will determined by a different group of investors at different times depending on which group performs better relative to others, therefore the risk-free rates exhibit a



(d1) $r(t) - r_f(t)$



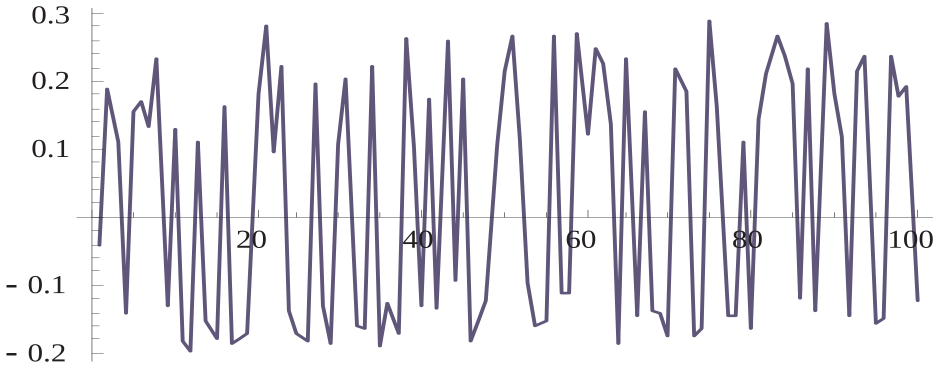
(d2) $r_f(t)$



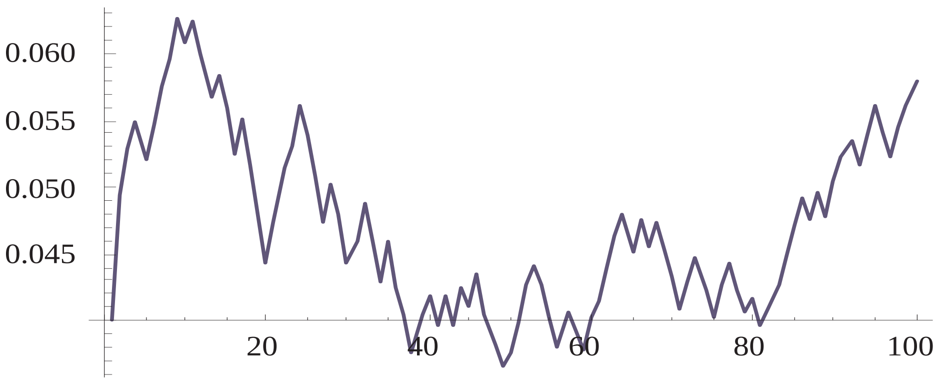
(d3) $w_i(t), i = 1, 2, \cdots, I$

FIGURE 6.1. Time series of the risk premium (d1), risk-free rate (d2) and individual agents' wealth share (d3).

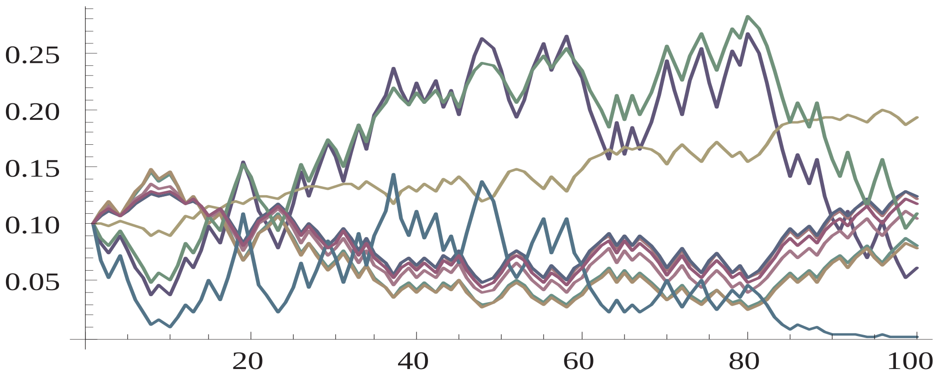
number of up and down trends. The risk-free rate in this case may not be a stationary process, even if it is, it could take a long for it to mean-revert back its mean.



(e1) $r(t) - r_f(t)$



(e2) $r_f(t)$



(e3) $w_i(t), i = 1, 2, \cdots, I$

FIGURE 6.2. Time series of the risk premium (e1), risk-free rate (e2) and individual agents' wealth share (e3).

Although not shown here, if we combine the previous two examples so that the agents revise their beliefs only at certain point of time, the exact time for revision could be depend on that particular agent's portfolio performance relative to others.

If he/she realizes that other agents' wealth shares have been increasing and his/hers decreasing, then he/she might decide to revise his/her belief. In this case, one would expect to see in the time series of the risk-free rate random fluctuations about a mean with temporary short periods of trends. Furthermore, investors' wealth share processes could form groups at different point of time, but diverge from each other later.

7. Heterogeneous Beliefs and Option prices

It is well know that CRR binomial model provides a good approximation for option prices under the Black-Scholes model. Although the model is derived under a non-arbitrage hedging argument, it is also the equilibrium price with a single log-utility investor maximizing his/her expected utility of terminal wealth. It is interesting to see what impact does heterogeneous beliefs have on the option prices in this model, whether heterogeneity can explain the implied volatility smile. Assume all investors believes in the CRR model, their beliefs regarding the future return of the stock in the up and downstate depend on their forecast of the volatility. More precisely, investor i's belief $\mathcal{B}_i(t)$ will be given by

$$u_i(t) = \exp(\sigma_i(t)\ \sqrt{\Delta t}) \qquad \text{and} \qquad d_i(t) = \exp(-\sigma_i(t)\ \sqrt{\Delta t}), \tag{7.1}$$

where $\Delta t = (T - t)/N$ is the time increment, with T and N denoting the *time to maturity* and *number of steps in the binomial lattice*. Then given the payoff function $f(\cdot, \cdot)$ of the option at maturity, it follows from Proposition 3.2 that the equilibrium price of the option at time t is

$$V(t, S(t)) = \frac{\mathbb{E}_t^{Q_m}(V(t+1, S(t+1)))}{1 + r_f(t+1)} = \frac{\mathbb{E}_t^m(Z(t+1)V(t+1, S(t+1)))}{1 + r_f(t+1)} \tag{7.2}$$

which can be re-written using the law of iterated expectations as

$$V(t, S(t)) = \mathbb{E}_t^m \left(V(T, S(T)) \prod_{s=t+1}^{T} \frac{Z(s)}{1 + r_f(s)} \right) = \mathbb{E}_t^m \left(f(T, S(T)) \prod_{s=t+1}^{T} \frac{Z^*(T)}{1 + r_f(s)} \right) \tag{7.3}$$

where $Z^*(T) = \prod_{s=t+1}^{T} Z(s)$ is a martingale under the market expectation. Under heterogeneous beliefs, the equilibrium interest rate $r_f(s)$ is stochastic and the martingale $Z^*(T)$ follows a different distribution for each sample path due to the fact that the market's belief $\mathcal{B}_m(s) = (u_m(s), d_m(s))$ is stochastic for $s \in [t+1, T]$.

However, it is almost impossible to predict the market's belief for the whole time interval, so what a typical investor would do is to assume $r_f(s) = \bar{r}_f$ is constant over the whole time interval and all investors have the same forecast for the volatility of future stock returns, that is $\sigma_i = \bar{\sigma}$ for all i and thus investors' belief will identically be

$$\bar{u} = \exp(\bar{\sigma} \ \sqrt{\Delta t}) \qquad \text{and} \qquad \bar{d} = \exp(-\bar{\sigma} \ \sqrt{\Delta t}) \tag{7.4}$$

for all time $t \in [t+1, T]$ and the option price would be

$$\bar{V}(t, S(t)) = \frac{\bar{\mathbb{E}}_t(Z(t+1)V(t+1))}{1 + r_f(t+1)} = \frac{1}{1 + \bar{r}_f}(\bar{q}_u \bar{V}(t+1, \bar{u}S(t)) + \bar{q}_d \bar{V}(t+1, \bar{d}S(t))) \tag{7.5}$$

The recursion in equation (7.5) has the explicit solution,

$$\bar{V}(t, S(t)) = \frac{1}{1 + \bar{r}_f^N} \sum_{l=0}^{N} C_l^N q_u^l q_d^{N-l} f(T, \bar{u}^l \bar{d}^{N-l} S(t)) \tag{7.6}$$

where $C_l^N \equiv \frac{N!}{l!(N-l)!}$ is one of the binomial coefficients, see Hoek and Elliot (2006). It was shown in Cox, Ross and Rubinstein (1979) that the formula in equation (7.6) approaches to the theoretical option price of the Black-Scholes formula. This prompts the question of how heterogeneous belief will impact on the price of the option. We now try to demonstrate this using the following numerical example.

Example 7.1. *Assume there are $I = 10$ investors in the market. The benchmark volatility is $\bar{\sigma} = 0.1654$. Assume individual investors' forecast of the volatility form a MPS around the benchmark. More precisely, $\sigma_i \overset{\text{iid}}{\sim} Unif(-\delta + \bar{\sigma}, \delta + \bar{\sigma})$. Investor i's belief $\mathcal{B}_i = (u_i, d_i)$ of the future stock return will be determined by equation (7.1) and the benchmark belief $\bar{\mathcal{B}} = (\bar{u}, \bar{d})$ will be calculated in a similar fashion using equation (7.4). Under the benchmark belief, the option prices can be found explicitly using equation (7.3) whereas under heterogeneous beliefs, we need to use Monte-Carlo simulation to evaluate the expectation in equation 7.3.*

In Example 7.1, investors' forecast of the volatility diverge uniformly from the benchmark volatility and one can view δ as the measure for divergence of opinion regarding stock volatility. This implies that investor i's belief $\mathcal{B}_i$ can be written in the following form in terms of the benchmark belief,

$$u_i = \exp\{\tilde{x}_i\sqrt{\Delta t}\}\ \bar{u} \qquad d_i = \exp\{-\tilde{x}_i\sqrt{\Delta t}\}\ \bar{d} \tag{7.7}$$

where $\tilde{x}_i \overset{\text{iid}}{\sim} Unif(-\delta, \delta)$. Equation (7.7) shows a little divergence of opinion in the volatility forecast can translate into a much greater divergence of opinion in the belief of future stock returns. Furthermore, we assume that investors do not revise their beliefs within the time interval $[t, T]$, therefore we can omit the time subscript.

Figure 7.1 shows the European call option prices with different strikes under the benchmark belief, heterogeneous beliefs and using the Black-Scholes formula. We used $N = 10$ number of steps in the binomial lattice and furthermore we use $100,000$ simulations to evaluate the option under heterogeneous beliefs. We need to utilize Monte-Carlo and binomial lattice together to evaluate the call option price under heterogeneous beliefs. The market's belief of the future stock return in the up and down state at each period is random and hence it depends on the distribution of investors' wealth shares at each period in the future.

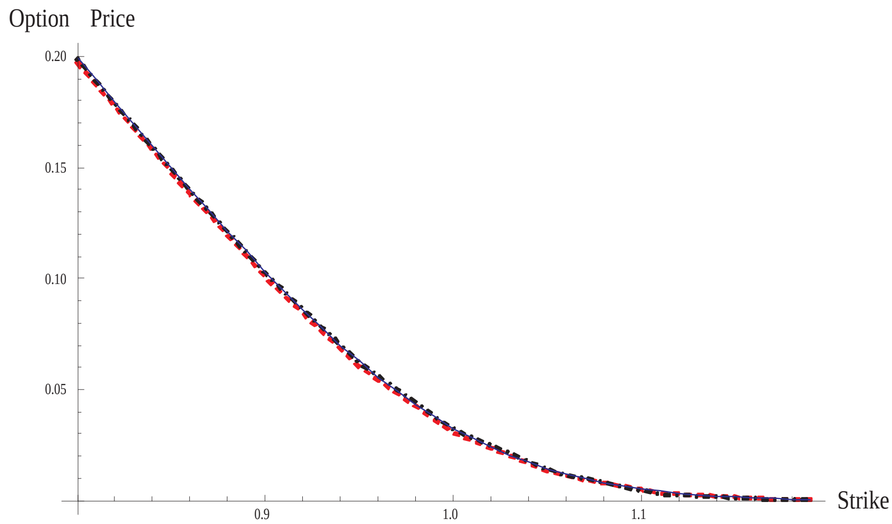


FIGURE 7.1. European call option prices with $\delta = 0.01$. The full line represents the option prices evaluated using the Black-Scholes formula, dashed line corresponds to option prices under the homogeneous binomial model and dot-dashed line corresponds to option prices under the heterogeneous binomial model.

Figure 7.1 shows that call prices evaluated using three different methods almost coincide. However the question remains what patterns the implied volatility exhibits when we use the option prices evaluated under heterogeneous belief as inputs to the Black-Scholes formula to back out the volatility parameter.

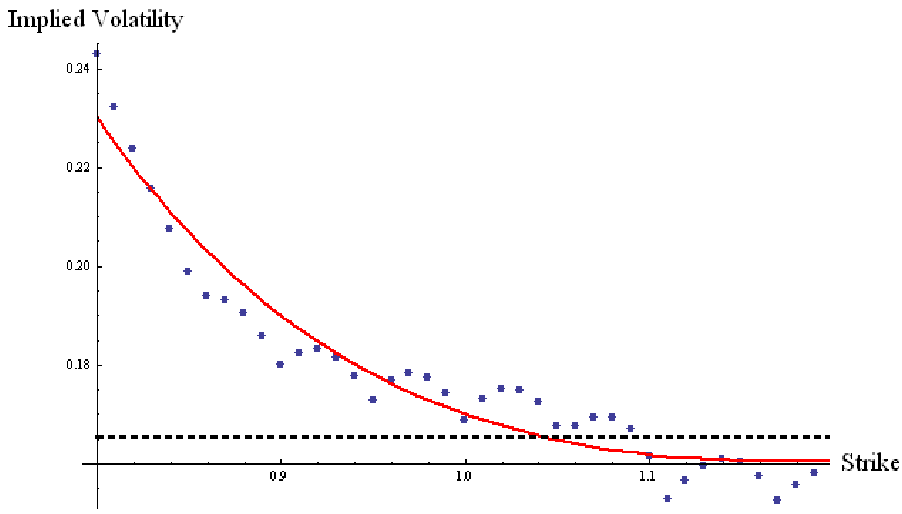


FIGURE 7.2. Implied volatility of the call option prices evaluated under heterogeneous beliefs with $\delta = 0.01$ (dots), an fitted function $f(z) = a + b/z + c/z^2$ with $a = 0.48$, $b = 0.44$ and $c = -0.75$ (full line). The dashed line is the benchmark volatility $\sigma = 0.1654$.

Figure 7.2 shows that call option prices under the heterogeneous binomial model exhibit a skew. The periodic humps in the implied volatility are due to the approximation error of the binomial lattice to the Black-Scholes continuous time model. This confirms the intuition that heterogeneity provides an explanation to the observed "volatility skew" observed in the implied volatility of options traded in the real market.

Now we try to give some intuitions for this result. Heterogeneity in investor's beliefs of the volatility places additional uncertainties on the stock price process, therefore the stock price is more volatile with heterogeneous beliefs than homogeneous beliefs under market expectation. As a result, the aggregate market demands a higher premium for selling a call or a put option, especially for deep ITM call and deep OTM put. Furthermore, to use this model to pricing options in practice, one would need to estimate how investor's forecast of the volatility σ is distributed, it seems not enough to just estimate the volatility from historical data.

8. CONCLUSION

In this paper, we provide an aggregation method of heterogeneous beliefs within a multi-period binomial lattice framework. The heterogeneity is characterized by heterogeneous beliefs regarding the return of the asset in the next period. Investors are bounded rational in the sense that they make optimal portfolio decisions based on their belief. To analyze the impact of heterogeneity, we introduce the concept of market consensus belief, which relate our heterogeneous market to an equivalent homogeneous market. We know that under market aggregation, the consensus belief is a wealth weighted harmonic mean of the heterogeneous beliefs. Through various numerical examples, we examined the impact of heterogenous beliefs on the equilibrium risk-free rate, the equity risk premium and option prices. The following results are obtained; (i) The market expects a lower (higher) future return when investor's opinions diverge regarding the future stock return in upstate (downstate), (ii) When there is more uncertainty regarding future stock return in the upstate than in the downstate, the expected equilibrium risk-free rate is reduced and equity risk-premium increased. This can partially explain the observed average equity premium and risk-free rate in the market. (iii) Dynamically, when investors revise their beliefs every period, their wealth shares evolve randomly over time and the equilibrium risk-free rate fluctuate about its mean. However, when investors hold on to their initial belief, wealth share processes of investors tend to form groups and equilibrium risk-free rate starts show trends and mean revert very slowly if it mean reverts at all. (iv) Heterogeneity is able to explain the observed implied volatility skew in equity options.

In summary, heterogeneity or divergence of opinions amongst investors introduce more uncertainty into the market. Consequently, the aggregate market demands a premium for compensation, thus the equity premium is higher than the homogeneous benchmark value. This leads to higher demand for the risk-free security which lowers the equilibrium risk-free rate. As for option prices, under heterogeneous beliefs, European call and put prices no longer approach the Black-Scholes theoretical option price. The aggregate market requires a higher premium than the benchmark option price, due to the fact that heterogeneity leads to more uncertainty about the option payoff at maturity. This produces a skewed implied volatility curve when one tries to back out the volatility parameter σ by using the observed option price as an input.

APPENDIX A. PROOF OF PROPOSITION 3.2

Inserting the expression for investors' optimal portfolio in equation (2.3) into the market clearing condition in equation (3.1) leads to

$$\sum_{i=1}^{I}(1+r_f(t+1))\frac{\tilde{u}_i(t+1)\ p+\tilde{d}_i(t+1)\ (1-p)}{-\tilde{u}_i(t+1)\ \tilde{d}_i(t+1)}\ W_i(t)=W_m(t) \tag{A.1}$$

On the one hand, if every investor has identical belief $\mathcal{B}_m(t)$ about the future return in the upstate and downstate respectively. i.e $\tilde{u}_i(t+1)=\tilde{u}_m(t+1)$ and $\tilde{d}_i(t+1)=\tilde{d}_m(t+1)$ for all i. Then it is obvious that equation (A.1) becomes

$$(1+r_f(t+1))\frac{\tilde{u}_m(t+1)\ p+\tilde{d}_m(t+1)\ (1-p)}{-\tilde{u}_m(t+1)\ \tilde{d}_m(t+1)}\ =1 \tag{A.2}$$

On the other hand, equation (A.1) can be re-written as

$$(1+r_f(t+1))\sum_{i=1}^{I}\frac{\tilde{u}_i(t+1)\ p+\tilde{d}_i(t+1)\ (1-p)}{-\tilde{u}_i(t+1)\ \tilde{d}_i(t+1)}\ \frac{W_i(t)}{W_m(t)}=1$$

$$\Rightarrow (1+r_f(t+1))\left[\sum_{i=1}^{I}p\ \frac{-w_i}{\tilde{d}_i(t+1)}+\sum_{i=1}^{I}(1-p)\ \frac{-w_i}{\tilde{u}_i(t+1)}\right]=1 \tag{A.3}$$

then using the definition in equation (3.4) and (3.5), we will obtain

$$(1+r_f(t+1))\left[\frac{p}{-\tilde{d}_m(t+1)}-\frac{1-p}{\tilde{u}_m(t+1)}\right] \tag{A.4}$$

which simplifies to the same relation as stated in equation (A.2). Therefore, $\mathcal{B}_m(t)=(u_m(t+1),\ d_m(t+1))$ is the consensus belief where $u_m(t+1)=\tilde{u}_m(t+1)+(1+r_f(t+1))$ and $d_m(t+1)=\tilde{d}_m(t+1)+(1+r_f(t+1))$ and $S(t)=W_m(t)$ will the common equilibrium price for the risky asset in both cases. The relation in expression (A.2) leads to the equilibrium risk-free rate in (3.6).

If we normalize the market wealth to 1, then it is clear that at time t

$$1 = S(t) = q_u(t)\ u(t+1) + q_d(t)\ d(t+1) \tag{A.5}$$

where $q_u(t)$ and $q_d(t)$ are the state prices for upstate and downstate respectively. Since under the market consensus belief, we have $u_(t+1) = u_m(t+1)$ and $d_(t+1) = d_m(t+1)$ and following from equation (A.2), one can see that state prices under the market consensus belief must be as specified in equations (3.8) and (3.9). Furthermore, the fact that q_u and q_d are positive and sum to 1 means that $\{q_u, q_d\}$ can be interpreted as a probability measure, we call this the *risk neutral probability measure* under the market consensus belief, because under this measure, the stock price at any time $t < T$ is just its future value at time $t+1$ discounted by the risk-free rate. This leads to the expression in equation (3.10) and if we want to evaluate the expectation under the original probability measure under the market consensus belief, the *Radon Nikodym derivative* must be defined as in equation (3.12). This completes the proof.

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